SUSPENSION BRIDGES WITH ELASTIC CHAINS

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The shape of the suspension cables of the suspension bridges, the corresponding equations to determine the tension at any point on the cables and the arcual length of the cables have been determined considering the suspended cables to be made up of elastic materials.

In the case of suspension bridges, the roadway is attached by suspension rods spaced usually at equal horizontal distances apart, to the two suspended cables of the bridge, the cables being properly anchored down at the ends. The shape of the suspended cables of the suspension bridges[1] approximates to a parabola of the form

$$x^2 = 2cy \tag{1}$$

where the axes of 'X' and 'Y' are respectively horizontal and vertical, the origin being at the lowest point of the curve and 'c' is a certain length of the horizontal span such that when it is multiplied with the load per unit length of the horizontal span gives the tension at the lowest point. Eqn. (1) as well as all other equations given in standard text book on Mechanics of Structures[2] now in use for the analysis of suspension bridges have been derived on the assumption that the cables of the suspension bridges are non-elastic. This assumption seems to be incorrect because in practice the cables of the suspension bridges are made up of steel wire ropes, which are definitely elastic materials. In the present analysis the elasticity of the cables has been taken into account and the corresponding equations have been derived.

EQVILIBRIUM OF THE STRINGS UNDER THE ACTION OF GIVEN FORCES

Let 'P' be any point (x,y) on the string whose arcual distance from a fixed point is 's'. Let 'Q' be a point very close to 'P', so that $PQ = s$ and hence 'Q' is the point $(x+\delta x, y+\delta y)$. Let 'T' and '$T+\delta T$' be the tension at 'P' and 'Q'. Let the forces per unit of mass of the string at 'P' be 'X' and 'Y', so that on the element PQ the components parallel to the axis are $m\delta sX$ and $m\delta sY$, where 'm' is mass per unit of length.

The tension at 'P' resolved parallel to the axis of 'X' is Tdx/ds and this is clearly equal to some function $f(s)$ of the arc 's', since it depends on the position of 'P'.

The tension at 'Q' resolved parallel to OX, is given by

$$f(s+\delta s) = f(s) + \delta s f'(s) + \dots\dots\dots\dots$$

$$= Tdx/ds + \delta s \times d(Tdx/ds)/ds + \dots\dots\dots\dots \tag{2}$$

Equating forces on PQ in the direction of the axis of 'X', we have

$$- T\frac{dx}{ds} + \left\{ T\frac{dx}{ds} + \delta s \frac{d}{ds}\left(T\frac{dx}{ds}\right) + \ldots\ldots\ldots\ldots \right\} + m\delta s\, X = 0$$

or

$$\delta s \frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX\,\delta s = 0 \qquad (3)$$

Similarly, resolving the forces parallel to the axis of 'Y' we get

$$\delta s \frac{d}{ds}\left(T\frac{dy}{ds}\right) + m\,Y\,\delta s = 0 \qquad (4)$$

Equations (3) and (4) are the equilibrium equations of the string under the given forces.

SUSPENSION BRIDGES WITH LINEAR ELASTICITY THEORY

Let (x,y) be the co-ordinates of a point whose arcual distance from the lowest point of the curve is 's' and let 'T' be the tension at that point. Let 's_0' be the unstretched length of this arc, so that

$$T = E\,\frac{ds - ds_0}{ds_0} \qquad (5)$$

Therefore

$$ds/ds_0 = 1 + T/E \qquad (6)$$

'E' being the Youngs constant. If 'c' be the length as defined earlier and 'w' is the load per unit length, then

$$T_0 = wc \qquad (7)$$

The external forces along 'X' axis is zero and along 'Y' axis is given by

$$mY\,\delta s = -\,w\,\delta x_0 \qquad (8)$$

as the string is loaded so that the weight on each element of it is proportional to the horizontal projection of that element. Hence the Eqns. (3) and (4) take the form

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0 \qquad (9)$$

and

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) = w\,\frac{\delta x_0}{\delta s} \qquad (10)$$

Equation (9) gives

$$T\,dx/ds = \text{a constant} = T_0 = wc \tag{11}$$

Equation (10) on re-arrangement gives

$$\frac{d}{dx}\left(T\ \frac{dy}{ds}\right) = w\ \frac{\delta x_0}{\delta x} \tag{12}$$

From Eqns, (11) and (12), we get

$$\frac{d^2y}{dx^2} = \frac{1}{c}\ \frac{\delta x_0}{\delta x} \tag{13}$$

Substituting the value of $\frac{\delta x_0}{\delta x}$ from Eqn. (6) in Eqn. (13) we get

$$\frac{d^2y}{dx^2} = \frac{1}{c}\left(1 + \frac{T}{E}\right)^{-1} \tag{14}$$

Substituting the value of 'T' as obtained from Eqn. (11) in Eqn. (14) and on simplification Eqn. (14) takes the form

$$\frac{d^2y}{dx^2} = \frac{1}{c}\left[1 + \frac{wc}{E}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}\right]^{-1} \tag{15}$$

Solution of Eqn. (15), with the boundary condition that when $x = 0$, the slope is zero, gives

$$y = \frac{x^2}{2c\left(1 + \frac{w\,c}{E}\right)} \tag{16}$$

which is the equation of the shape the elastic cable of the suspension bridge will take up.

Eqn. (16) indicates that the slope of the cables of the suspension bridge according to the linear elasticity theory is a parabola of latus rectum $2c\,(1 + wc/E)$. Comparing Eqns. (16) and (1) it is clear that the latus rectum of the parabola given by Eqn. (16) (the shape, the cables take based on linear elasticity theory) is greater than the latus rectum of the parabola given by Eqn. (1) (the shape, the non-elastic cables take up).

Eqn. (16) compared with Eqn. (1) indicates that the maximum sag of the cables of the suspension bridges according to linear elasticity theory will be smaller than that of non-elastic cable suspension bridges. This indicates that the support-distance of the suspension bridge remaining constant. The total arcual length of the suspension cables will decrease when the elsaticity effect is considered and the maximum tension at the supports to which cables will be subjected will increase due to the elasticity of the cables. Eqn. (1) can also be obtained as a particular case from Eqn. (16) when $E = \infty$ which indicates that the material is non-elastic. Now from Eqn. (11) the tension at any point on the cable is given by

$$T = wc \frac{ds}{dx} = wc \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} \tag{17}$$

Substituting the value of $\frac{dy}{dx}$ from Eqn. (16) in Eqn. (17) we get

$$T = wc \left\{ 1 + \frac{x^2}{c^2 \left(1 + \frac{wc}{E} \right)^2} \right\}^{\frac{1}{2}} \tag{18}$$

which on simplification gives

$$T = \frac{\omega}{\left(1 + \frac{wc}{E} \right)} \left\{ x^2 + c^2 \left(1 + \frac{wc}{E} \right)^2 \right\}^{\frac{1}{2}} \tag{19}$$

Eqn. (19) gives the tension at any point on the cables in which the only variable factor is 'x'. Hence 'T' is maximum when $x = \frac{l}{2}$, i.e., at the supports and is given by

$$T_{max.} = \frac{\omega}{\left(1 + \frac{wc}{E} \right)} \left\{ \frac{l^2}{4} + c^2 \left(1 + \frac{wc}{E} \right)^2 \right\}^{\frac{1}{2}} \tag{20}$$

and 'T' is minimum when $x = 0$, i.e., at the lowest point of the parabola and is given by

$$T_{min.} = w\,c$$

as given in Eqn. (11). Comparing Eqn. (19) and the corresponding equation for the cables without considering the elastic effect, which can also be obtained as a particular case of Eqn. (19) putting $E = \infty$ for non-elastic material, it is clear that Eqn. (19) gives higher value which is in line with the discussion given after Eqn. (16).

To calculate the total arcual length of the cable, the elementary distance 'PQ' is given by

$$ds = \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} dx \tag{21}$$

Substituting the value of dy/dx from Eqn. (16) in Eqn. (21)

$$ds = \left[1 + \left\{ \frac{x}{c\left(1 + \frac{wc}{E} \right)} \right\}^2 \right]^{\frac{1}{2}} dx \tag{22}$$

As in the case of suspension bridges the maximum sag is very small compared to the span of the bridge, the slope of the suspended cables is very small. Hence the factor $\dfrac{x}{c\left(1+\dfrac{wc}{E}\right)}$ can be taken as a fraction. Expanding by the binomial theorem and neglecting the terms higher than the second powers of 'x', Eqn. (22) reduces to

$$ds = \left\{1 + \tfrac{1}{2}\frac{x^2}{c^2\left(1+\dfrac{wc}{E}\right)^2}\right\} dx \tag{23}$$

The total arcual length of the cable is given by,

$$L = 2\int_0^{l/2}\left[1 + \tfrac{1}{2}\frac{x^2}{c^2\left(1+\dfrac{wc}{E}\right)^2}\right] dx \tag{24}$$

which on integration gives

$$L = l + \frac{l^3}{24\,c^2\left(1+\dfrac{wc}{E}\right)^2} \tag{25}$$

If 'y_c' is the maximum sag of the cables under consideration substituting the value of the term $\{c\,(1+wc/E)\}$ of the Eqn. (25) in terms of 'y_c' as obtained from Eqn. (16), in Eqn. (25) reduces to

$$L = l + \frac{8}{3} \times \frac{y^2_c}{l} \tag{26}$$

which is exactly of the same form that is obtained without considering the elastic effect[2]. But in Eqn. (26) the value of 'y' is to be obtained from Eqn. (16).

As it has been stated earlier that the maximum sag of the elastic cables is smaller than that of the non-elastic cables, the total arcual length of the elastic cables will be smaller than that obtained for non-elastic cables which is evident from the Eqn. (26).

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