PROBLEM IN LIFE TESTING WITH CHANGING FAILURE RATE

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In life test experiments, a situation may arise when the items are put on test at different times and at one particular instant; we know only the time of failure or the time since it is on test if it has not failed upto that instant. It has been assumed that the probability density function of the life of an item and hence its failure rate changes after an item is on test for certain time. The method of maximum likelihood has been used to estimate the failure rates in the case when the life of an item follows the exponential distribution,

Bartholomew[1] has discussed the estimation of mean life of an item in a life test experiment when the items are placed on test at different times so that at one particular instant either the life of an item is known (if it has already failed) or the time that has elapsed since the item was placed on test, is known. He assumed that the life time distribution of an item remains the same throughout the experiment. However, in services certain stores and equipments are subjected to regular check up even though they are functioning normally. When such items are placed on life test, after sometime the items that have not failed are checked up and overhauled repairing the minor defects. This, naturally, changes the life time distribution of the items and consequently the failure rate also changes. In this paper it has been assumed that this change takes place only once before the test is terminated (after certain stipulated time, recorded from the beginning of the experiment). The case where several changes take place is being studied.

Let n items be placed on test at different times, the life time of each following the expotial distribution with failure rate λ_1. The failure rate is defined as the probability of instantaneous failure of the item at a time provided the item has not failed upto that time. If $f(t)$ reprsents the probability density function of the life of an item and $F(t)$ represents the distribution function then failure rate is given by[2]

$$\frac{f(t)}{1-F(t)}$$

It can be easily seen that the failure rate is constant in the case of exponential distribution.

After certain, known time T_o, let the life time of each item follow the exponential distribution with failure rate λ_2 and let the test be finally terminated at known time T_1. Then the composite probability density function $p(t)$ and distribution function $P(t)$ are given by[3]

$$p(t) = \begin{cases} p_1(t) = \lambda_1 \; exp \; (-\lambda_1 t) \\ p_2(t) = \lambda_2 \; exp \; [-\lambda_1 T_o - \lambda_2 \; (t - T_o)] \end{cases}$$

$$Q(t) = 1 - P(t) = \begin{cases} P_1(t) = exp \; (-\lambda_1 t) \\ P_2(t) = exp \; [-\lambda_2 t - (\lambda_1 - \lambda_2) \; T_o] \end{cases}$$

LIKELIHOOD FUNCTION

During the experiment, at any particular instant $t > T_o$, let the time that has elapsed since the *ith* item was placed on test be T_i and let the life of this item be t_i, which will be known only if $t_i \leqslant T_i$. Any given sample will, therefore, consist of the quantities

$$T_1, T_2, \ldots\ldots\ldots\ldots T_n$$

and a certain number of completed lives. The completed lives could be before time T_o or after time T_o. However, it is assumed that

$$\underset{\Sigma}{Min} \left\{ T_i \right\} > T_o$$

Then the likelihood function of the sample is

$$P(S) = \text{Const} \prod_{i=1}^{n} \left\{ p_1 (t_i) \right\}^{a_i} \left[Q_{2}^{1-a'_i} (T_i) \left\{ p_2 (t_i) \right\}^{a'_i} \right]^{1-a_i}$$

where

$a_i = 1$, if the item placed time T_i ago has failed by time T_o,

$= 0$, if the item placed time T_i ago has not failed by time T_o

$a_i' = 1$, if the item placed time T_i ago has failed after time T_o.

$= 0$, if the item placed time T_i ago has not failed.

ESTIMATES FOR λ_1 AND λ_2

Taking the logarithm of the likelihood function and simplifying we get

$\log P(S) = L$ (say)

$$= \sum_{i=1}^{n} \left[\begin{array}{l} a_i \; (\log \; \lambda_1 - \lambda_1 \; t_i) \\ + (1 - a_i) \; \{ \; (1 - a'_i) \; (- \lambda_2 \overline{T_i - \lambda_1 - \lambda_2 T_o} \;) \\ \qquad + a'_i \; (\log \lambda_2 - \lambda_2 \; \overline{t_i - T_o} - \lambda_1 \; T_o \;) \; \} \end{array} \right]$$

Differentiating with respect to λ_1 and equating to zero we get

$$\sum_{i=1}^{n} \left\{ a_i \left(\frac{1}{\lambda_1} - t_i \right) + (1 - a_i) \; (- T_o) \right\} = 0$$

which gives the estimate for λ_1 as

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i \; t_i + T_o \sum_{i=}^{n} (1 - a_i)}$$

Differentiating log $P(S)$ with respect to λ_2 and equating to zero we get

$$\sum_{i=1}^{n}\left\{ (1-a_i)\,\overline{(1-\acute{a}_i)}\,(-T_i+T_o) + (1-a_i)\,\acute{a}_i\left(\frac{1}{\lambda_2}-t_i+T_o\right)\right\}=0$$

which gives the estimate for λ_2 as

$$\hat{\lambda}_2 = \frac{\sum_{i=1}^{n} \acute{a}_i\,(1-a_i)}{\left\{\sum_{i=1}^{n}(1-a_i)(1-\acute{a}_i)\,T_i + \sum_{i=1}^{n}(1-a_i)\,\acute{a}_i\,t_i - T_0\sum_{i=1}^{n}(1-a_i\right\}}$$

It can be easily seen that if $T_o = 0$ which implies that $a_i = 0$ our estimate is the same as obtained by Bartholomew[1]. The asymptotic variance-covariance matrix of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ is given as

$$\begin{Bmatrix} -E\left(\dfrac{\partial^2 L}{\partial \lambda_1^2}\right) & -E\left(\dfrac{\partial^2 L}{\partial\lambda_1\partial\lambda_2}\right) \\ -E\left(\dfrac{\partial^2 L}{\partial\lambda_1\partial\lambda_2}\right) & -E\left(\dfrac{\partial^2 L}{\partial\lambda^2_2}\right)\end{Bmatrix}^{-1} = \begin{Bmatrix} V(\hat{\lambda}_1) & \text{Cov}(\hat{\lambda}_1, \hat{\lambda}_2) \\ \text{Cov}(\hat{\lambda}_1, \hat{\lambda}_2) & V(\hat{\lambda}_2)\end{Bmatrix}$$

Hence it can be easily seen that

$$V(\hat{\lambda}_1) = \frac{\lambda_1^2}{n\,P_1\,(T_o)}$$

$$V(\hat{\lambda}_2) = \frac{\lambda_2^2}{Q_1\,(T_o)\sum_{i=1}^{n} P_2\,(T_i)}$$

$$\text{Cov}\,(\hat{\lambda}_1, \hat{\lambda}_2) = 0$$

BIAS OF THE ESTIMATES

Instead of considering the bias of estimates $\hat{\lambda}_1$ and $\hat{\lambda}_2$ the bias of $\frac{1}{\hat{\lambda}_1}$ and $\frac{1}{\hat{\lambda}_2}$ is separately considered.

Bias of $\frac{1}{\hat{\lambda}_1}$

Let K denotes the number of items that have failed by time T_0. so that

$$K = \sum_{i=1}^{n} a_i$$

Now

$$\text{Cov.}\left(K, \frac{1}{\hat{\lambda}_1}\right) = E\left(\frac{K}{\hat{\lambda}_1}\right) - E(K).\, E\left(\frac{1}{\hat{\lambda}_1}\right)$$

Therefore

$$E\left(\frac{1}{\hat{\lambda}_1}\right) = \frac{E\left(\frac{K}{\hat{\lambda}_1}\right)}{E(K)} - \frac{\text{Cov.}\left(K\ \frac{1}{\hat{\lambda}_1}\right)}{E(K)}$$

Now $\frac{K}{\hat{\lambda}_1}$ may be regarded as the sum of n variates each coming from the population :

$$g(t) = \lambda_1 \exp.(-\lambda_1 t)\ ; \quad 0 \leqslant t < T_0$$
$$= \exp.(-\lambda_1 T_0)\ ; \quad t = T_0$$

Thus

$$E(t) = \int_0^{T_0} t\, g(t)\, dt = \frac{1}{\lambda_1} P_1(T_0)$$

and therefore,

$$E\left(\frac{K}{\hat{\lambda}_1}\right) = \frac{n\, P_1(T_0)}{\lambda_1}$$

Moreover, since $E(K) = nP_1(T_0)$, we get

$$E\left(\frac{1}{\hat{\lambda}_1}\right) = \frac{1}{\lambda_1} - \frac{\text{Cov.}\left(K, \frac{1}{\hat{\lambda}_1}\right)}{n\, P_1(T_0)}$$

Now, it can be, intuitively, seen that Cov. $\left(K, \frac{1}{\hat{\lambda}_1}\right)$ will always be negative and hence it can be concluded that $\frac{1}{\hat{\lambda}_1}$ will always overestimate $\frac{1}{\lambda_1}$ and the bias of estimate $\frac{1}{\hat{\lambda}_1}$ is

$$\left| \frac{\text{Cov.}\left(K, \frac{1}{\hat{\lambda}_1}\right)}{n\, P_1(T_0)} \right|$$

By using the inequality

$$\left(K, \frac{1}{\hat{\lambda}_1} \right) \leqslant \left\{ \text{Var.}\,(K).\ \text{Var.} \left(\frac{1}{\hat{\lambda}_1} \right) \right\}^{\frac{1}{2}}$$

it can be easily shown that

$$\text{Cov.} \left(K, \frac{1}{\hat{\lambda}_1} \right) \leqslant \frac{1}{\lambda_1}$$

and hence, for the magnitude of the bias of estimate $\frac{1}{\hat{\lambda}_1}$ we get.

$$\left| E \left(\frac{1}{\hat{\lambda}_1} \right) - \frac{1}{\lambda_1} \right| \leqslant \frac{1}{n\,\lambda_1\,P_1\,(T_0)}$$

Bais of $\frac{1}{\hat{\lambda}_2}$

Let K' denote the number of items that failed after time T_0 but before time T_i so that $K' = \sum_{i=1}^{n} (1 - a_i)\, a'_i$ and $E\,(K') = \sum_{i=1}^{n} P_2\,(T_i)$. In this case $\left(\frac{K'}{\hat{\lambda}_2} \right)$ may be regarded as the sum of n variates each coming from the population :

$$h\,(t) = \lambda_2 \exp. \left[-\lambda_1 T_0 - \lambda_2\,(t - T_0) \right];\ T_0 < t < T_i$$

$$= \exp. \left[-\lambda_1 T_0 - \lambda_2\,(T_i - T_0) \right];\ t = T_i$$

so that

$$E\,(t) = \int_{T_0}^{T_i} t\,h\,(t)\,dt = \frac{P_2\,(T_i)}{\lambda_2} + T_0\,Q_1\,(T_0) - \frac{P_0\,(T_0)}{\lambda_2}$$

and

$$E \left(\frac{1}{\hat{\lambda}_2} \right) = \frac{1}{\lambda_2} + h.\ \frac{T_0\,Q_1\,(T_0) - \dfrac{P_1\,(T_0)}{\lambda_2}}{\sum_{i=1}^{n} P_2\,(T_i)} - \frac{\text{Cov.} \left(K', \dfrac{1}{\hat{\lambda}_2} \right)}{\sum_{i=1}^{n} P_2\,(T_i)}$$

Hence for the magnitude of the bias of the estimate $\frac{1}{\hat{\lambda}_2}$ we get

$$\left| \left(\frac{1}{\hat{\lambda}_2} \right) - \frac{1}{\lambda_2} \right| \leqslant h \frac{\left| T_0 \, Q_1 (T_0) - \frac{P_1 (T_0)}{\lambda_2} \right|}{\sum_{i=1}^{n} P_2 (T_i)} + \frac{1}{K \sum_{i=1}^{n} P_2 (T_i)}$$

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