FLOW PARAMETERS BEHIND AN UNSTEADY CURVED SHOCK WAVE IN AN IDEAL GAS WITH HEAT ADDITION

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This paper discusses the flows immediately behind and in front of a two dimensional unsteady curved shock wave in an ideal gas with heat addition. Method evolved by Thomas has been followed to obtain formulas for the determination of the gradients of velocity components, pressure and density behind the shock when the flow in front is known.

JUMP CONDITIONS

The equations governing the flow are:

$$\frac{D\rho}{Dt} + \rho_{,\alpha}\left(U_\alpha - G\,\nu_\alpha\right) + \rho U_{\alpha,\alpha} = 0 \tag{1}$$

$$\rho\,\frac{D\,U_\alpha}{Dt} + \rho\,U_{\alpha,\beta}\left(U_\beta - G\,\nu_\beta\right) + p_{,\alpha} = 0 \tag{2}$$

$$\frac{Dp}{Dt} - \rho\,\frac{DU_\alpha}{Dt}\left(U_\alpha - G\,\nu_\alpha\right) + c^2\rho\,U_{\alpha,\alpha} - \rho\,U_{\beta,\alpha}\left(U_\beta - G\,\nu_\beta\right)\left(U_\alpha - G\,\nu_\alpha\right) = 0 \tag{3}$$

where G denotes any speed, ν_α any direction and

$$\frac{Df}{Dt} \underset{=}{\overset{def}{}} \frac{\partial f}{\partial t} + G\,f_{,\alpha}\,\nu_\alpha$$

such that $G\,\nu_\alpha$ is taken as the velocity of propagation of the shock. The jump conditions[1] for this case become

$$\left[\,U_\alpha\,\right] = -\,\frac{S}{1+S}\left(U_{1n} - G\right)\nu_\alpha \tag{4}$$

$$\left[\,p\,\right] = \frac{S}{1+S}\,\rho_1\left(U_{1n} - G\right)^2 \tag{5}$$

where

$$S \overset{def.}{=\!=} \frac{\sigma}{(\gamma+1)\,\rho_1\,(U_{1n} - G)^2 - \sigma} = \frac{[\rho]}{\rho_1} \tag{6}$$

$$\sigma = \rho_1\,(U_{1n} - G)^2 - \gamma p_1 - \delta \tag{7}$$

$$\delta^2 = \left\{\gamma p_1 - \rho_1\,(U_{1n} - G)^2\right\}^2 - 2\,h\,\rho_1^2\,(\gamma^2 - 1)\,(U_{1n} - G)^2 \tag{8}$$

Long brackets denote the difference in values of the quantity enclosed between the two sides of the shock. Subscript '1' has been used to denote quantities in front of the shock; similarly quantities without subscripts denote values behind the shock. Quantity h

which has been assumed to be constant[2], is the heat emitted per unit mass by a hot plate placed just in front of the shock. Summation convention and common notations of tensor calculus, wherever necessary, have been used.

DIFFERENTIATION OF THE SHOCK RELATIONS

By differentiating the relations (4), (5) and (6) *w.r.t.* the arc length s (measured along the shock), we get

$$U_{\alpha,\beta}\lambda^{\beta} = U_{1\alpha,\beta}\lambda^{\beta} - \left\{ \frac{S}{1+S}(U_{1n}-G)\nu_{\alpha} \right\}_{,\beta} \lambda^{\beta} = A_{\alpha} \qquad (9)$$

$$p_{,\beta}\lambda^{\beta} = p_{1,\beta}\lambda^{\beta} + \left\{ \frac{S}{1+S}\rho_1 (U_{1n}-G)^2 \right\}_{,\beta} \lambda^{\beta} = B \qquad (10)$$

$$\rho_{,\beta}\lambda^{\beta} = \rho_{1,\beta}\lambda^{\beta} + \left\{ S\rho_1 \right\}_{,\beta} \lambda^{\beta} = C \qquad (11)$$

where λ is the unit tangent vector to the shock and is so directed that the vectors λ, ν form a right handed system. The quantities A_{α}, B and C are expressible as functions of ρ_1, p_1, U_{1n}, G, ν_{α}, S and their derivatives[3]. A quantity so expressed will be said to have been effectively calculated. From the set of six equations (1), (2), (3), (9), (10) and (11) we determine the partial derivatives of the velocity components U_{α}, density ρ and pressure p immediately behind the shock. For this, we first eliminate $p_{,\alpha}$ to obtain the following equations

$$\rho_{,\beta}\lambda^{\beta} = C \qquad (12)$$

$$\rho_{,\alpha}(U_{\alpha} - G\nu_{\alpha}) + \rho U_{\alpha,\alpha} = -\frac{D\rho}{Dt} \qquad (13)$$

$$U_{\alpha,\beta}\lambda^{\beta} = A_{\alpha} \qquad (14)$$

$$\frac{DU\alpha}{Dt}\lambda^{\alpha} + U_{\alpha,\beta}(U_{\beta} - G\nu_{\beta})\lambda^{\alpha} = -\frac{B}{\rho} \qquad (15)$$

$$\frac{Dp}{Dt} - \rho\frac{DU\alpha}{Dt}(U_{\alpha} - G\nu_{\alpha}) + c^2\rho U_{\alpha,\alpha} - \rho U_{\beta,\alpha}(U_{\beta} - G\nu_{\beta})(U_{\alpha} - G\nu_{\alpha}) = 0 \qquad (16)$$

From (14), (15) and (16) we can determine $U_{\alpha,\beta}$ and then from (12) and (13) $\rho_{,\beta}$ can be found. (2) then yields the value of $p_{,\alpha}$. To achieve this, we define a matrix $\|\xi_i^{\alpha}\|$ as done by Thomas[4], such that

$$\|\xi_i^{\alpha}\| = \begin{Vmatrix} \xi_1^1 & \xi_1^2 \\ \xi_2^1 & \xi_2^2 \end{Vmatrix} = \begin{Vmatrix} \lambda^1 & \lambda^2 \\ (U_1 - G\nu_1) & (U_2 - G\nu_2) \end{Vmatrix}$$

The value of the determinant $|\xi_i^{\alpha}|$ is easily seen to be $(U_n - G)$ where $(U_n - G)$ now represents the normal component of the relative velocity immediately behind the shock. Again, let us define quantities ζ^i_{α} as :

$$\overset{i}{\zeta}_{\alpha} \overset{def}{=} \frac{\text{Cofactor of } \xi_i^{\alpha} \text{ in } |\xi_i^{\alpha}|}{|\xi_i^{\alpha}|}$$

So that

$$\| \zeta^i{}_\alpha \| = \left\| \begin{matrix} \zeta_1{}^1 & \zeta_1{}^2 \\ \zeta_2{}^1 & \zeta^2{}_2 \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{U_2 - G\nu_2}{U_n - G} & -\dfrac{\lambda^2}{U_n - G} \\ -\dfrac{U_1 - G\nu_1}{U_n - G} & \dfrac{\lambda^1}{U_n - G} \end{matrix} \right\|$$

Considering the expressions $U_{\alpha,\beta}\, \xi_i{}^\alpha\, \xi_j{}^\beta$, we have

$$U_{\alpha,\beta}\, \xi_1{}^\alpha\, \xi_1{}^\beta = U_{\alpha,\beta}\, \lambda^\alpha \lambda^\beta = A\alpha\lambda^\alpha = A_{11} \tag{17}$$

$$U_{\alpha,\beta}\, \xi_1{}^\alpha\, \xi_2{}^\beta = U_{\alpha,\beta}\, \lambda^\alpha\, (U_\beta - G\nu_\beta) = -\frac{B}{\rho} - \frac{DU_\alpha}{Dt}\, \lambda^\alpha = A_{12} \tag{18}$$

$$U_{\alpha,\beta}\, \xi_2{}^\alpha\, \xi_1{}^\beta = U_{\alpha,\beta}\, (U_\beta - G\nu_\alpha)\, \lambda^\beta = A_\alpha\, (U_\alpha - G\nu_\alpha) = A_{21} \tag{19}$$

$$U_{\alpha,\beta}\, \xi_2{}^\alpha\, \xi_2{}^\beta = U_{\alpha,\beta}\, (U_\beta - G\nu\alpha)\, (U_\beta - G\nu_\beta) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{DU\alpha}{Dt}\, (U\alpha - G\nu\alpha) + c^2\, U\alpha,\alpha = A_{22} \tag{20}$$

where any quantity A_{ij} is defined by the quantities immediately preceding it in the above relations. We have

$$U_{\alpha,\beta}\, \xi_i{}^\alpha\, \xi_j{}^\beta = A_{ij} \tag{21}$$

From (21) and the matrix above, we find

$$U_{\sigma,\tau} = A_{ij}\, \zeta^i{}_\sigma\, \zeta^j{}_\tau \tag{22}$$

Now, from the relations (17), (18), and (19), the quantities A_{11}, A_{12} and A_{21} are effectively calculated ; but since A_{22} is given in terms of $U\alpha,\alpha$ by (20), it still remains to be determined. For this we write, as a consequence of (22).

$$\begin{aligned} U\alpha,\alpha &= A_{ij}\, \zeta^i{}_\alpha\, \zeta^j{}_\alpha \\ &= A_{11}\, \zeta^1{}_\alpha\, \zeta^1{}_\alpha + (A_{12} + A_{21})\, \zeta^1{}_\alpha\, \zeta^2{}_\alpha + A_{22}\, \zeta^2{}_\alpha\, \zeta^2{}_\alpha \end{aligned} \tag{23}$$

But,

$$\zeta^1{}_\alpha\, \zeta^1{}_\alpha = \left(\frac{U_2 - G\nu_2}{U_n - G}\right)^2 + \left(\frac{U_1 - G\nu_1}{U_n - G}\right)^2 = \left(\frac{U\alpha - G\nu\alpha}{U_n - G}\right)^2$$

$$\zeta^1{}_\alpha\, \zeta^2{}_\alpha = \frac{-(U_2 - G\nu_2)\lambda^2}{(U_n - G)^2} - \frac{(U_1 - G\nu_1)\lambda^1}{(U_n - G)^2} = -\frac{(U\alpha - G\nu_\alpha)\lambda^\alpha}{(U_n - G)^2}$$

$$\zeta^2{}_\alpha\, \zeta^2{}_\alpha = \left(\frac{-\lambda^2}{U_n - G}\right)^2 + \left(\frac{\lambda^1}{U_n - G}\right)^2 = \frac{1}{(U_n - G)^2}$$

Substituting these values in (23), we get from (20);

$$A_{22} = \frac{c^2}{\rho\{(U_n - G)^2 - c^2\}}\Big[\rho A_\alpha \lambda^\alpha (U_\beta - G\nu_\beta)(U_\beta - G\nu_\beta) + U_t \Big\{ B - \rho A_\beta (U_\beta - G\nu_\beta) + \rho \frac{DU^\beta}{Dt}\lambda^\beta \Big\}\Big] + \frac{(U_n - G)^2}{\rho\{(U_n - G)^2 - c^2\}}\Big\{\frac{Dp}{Dt} - \rho\frac{DU^\alpha}{Dt}(U\alpha - G\nu\alpha)\Big\} \tag{24}$$

where U_t is the tangential component of the velocity along the shock line. Hence A_{22} is also found out. (12) and (13) can be written in the form $\rho_{,\beta}\ \xi_i^{\beta} = a_i$ where $a_1 = C$ and $a_2 = -\dfrac{D\rho}{Dt} - \rho\, U_{\alpha,\alpha}$. Hence,

$$\rho_{,\sigma} = a_i\ \zeta^i{}_{\sigma} \tag{25}$$

whence the quantities $\rho_{,\sigma}$ are effectively calculated. Finally, from (2), the derivatives $p_{,\alpha}$ are calculated with the help of the equations :

$$\begin{aligned} p_{,\alpha} &= -\rho \frac{DU_\alpha}{Dt} - \rho U_{\alpha,\beta} (U_\beta - G\nu_\beta) \\ &= -\rho \frac{DU_\alpha}{Ut} - \rho A_{i2}\ \zeta^i{}_\alpha \end{aligned} \tag{26}$$

VALUE OF THE INVARIANTS a_i AND A_{ij}

For uniform flow in region 1, we have

$$A_\alpha = -\left\{ \frac{S}{1+S} (U_{1n} - G)\ \nu_\alpha \right\}_{,\beta} \lambda^\beta \tag{27}$$

$$B = \left\{ \frac{S}{1+S}\ \rho_1 (U_{1n} - G)^2 \right\}_{,\beta} \lambda^\beta \tag{28}$$

$$C = \left\{ S\rho_1 \right\}_{,\beta} \lambda^\beta \tag{29}$$

Hence

$$A_\alpha = -\frac{B\nu_\alpha}{\rho_1 (U_1{}^n - G)} + \frac{[\rho]}{\rho} \left\{ k\lambda^\alpha (U_{1n} - G) - \nu^\alpha (kU_t + G_{,\beta}\lambda^\beta) \right\} \tag{30}$$

$$B = \frac{2\rho_1 (U_{1n} - G)(U_t\ k + G_{,\alpha}\lambda^\alpha)}{(\gamma+1)\,\delta} \left\{ \sigma - h\rho_1 (\gamma^2 - 1) \right\} \tag{31}$$

$$C = \frac{(U_{1n} - G)(U_t\ k + G_{,\alpha}\lambda^\alpha)}{\delta} [\rho] \left\{ (\gamma+1)\rho_1 - (\gamma-1)\rho \right\} \tag{32}$$

where k is the curvature of the shock at the point considered. Now (17)—(20), (24) and (30)—(32) can be used to calculate the quantities a_i and A_{ij} as follows :

$$A_{11} = kS\ (U_n - G) \tag{33}$$

$$A_{12} = \frac{-2(U_n - G)(U_t\ k + G_{,\alpha}\lambda^\alpha)}{(\gamma+1)\ \delta} \left\{ \sigma - h\rho_1 (\gamma^2 - 1) \right\} - \frac{DU_\alpha}{Dt} \lambda^\alpha \tag{34}$$

$$A_{21} = A_{12} + \frac{DU_\alpha}{Dt}\lambda^\alpha + \frac{[\rho]}{\rho}(U_n - G)\ \{ S\ kU_t - G_{,\alpha}\lambda^\alpha \} \tag{35}$$

$$A_{22} = \frac{c^2}{(U_n - G)^2 - c^2} \left\{ kS\,(U_n - G)\,v^2 - U_t\,(A_{12} + A_{21}) \right\} +$$

$$\frac{(U_n - G)^2}{(U_n - G)^2 - c^2} \left\{ \frac{Dp}{Dt} - \rho \frac{DU_\alpha}{Dt}\,(U_\alpha - G\nu_\alpha) \right\} \tag{36}$$

where v is the magnitude of the relative velocity behind the shock. It can be easily verified that the above relations reduce to the results obtained by (*i*) Thomas[4] in the case of stationary flow with no heat addition and (*ii*) Kanwal[5] when the magnetic field is not taken into account. Using the relations $a_1 = C$ and $a_2 = -\frac{D\rho}{Dt} - \rho U_{\alpha,\alpha}$ with $U_{\alpha;\alpha}$ given by (23) we have

$$a_1 = \frac{(U_{1n} - G)\,(U_t\,k + G\nu_\alpha\,\lambda^\alpha)}{\delta}\,[\rho]\,\{(\gamma + 1)\,\rho_1 - (\gamma - 1)\,\rho\} \tag{37}$$

$$a_2 = \frac{-\rho}{(U_n - G)^2 - c^2} \left\{ kS\,(U_n - G)\,v^2 - U_t\,(A_{12} + A_{21}) + \frac{Dp}{Dt} - \right.$$

$$\left. \rho \frac{DU_\alpha}{Dt}(U_\alpha - G\nu_\alpha) \right\} - \frac{D\rho}{Dt} \tag{38}$$

VARIATION OF VELOCITY, DENSITY AND PRESSURE ALONG THE SHOCK

We have

$$\frac{dU_\sigma}{ds} = U_{\alpha,\beta}\,\lambda^\beta = A_{ij}\,\zeta^i{}_\alpha\,\zeta^j{}_\beta\,\xi_1{}^\beta = A_{i1}\,\zeta^i{}_\alpha \tag{39}$$

$$\frac{d\rho}{ds} = \rho_{;\alpha}\,\lambda^\alpha = a_i\,\zeta^i{}_\alpha\,\xi_1{}^\alpha = a_1 \tag{40}$$

$$\frac{dp}{ds} = p_{;\alpha}\,\lambda^\alpha = -\rho A_{i2}\,\zeta^i{}_\alpha\,\xi_1{}^\alpha - \rho\frac{DU_\alpha}{Dt}\,\lambda^\alpha \tag{41}$$

(39) gives the value of the derivative $\frac{dv}{ds}$. When the flow in region 1 is uniform, the expressions A_{i1}, a_1 and A_{12} in the above relations can be taken from (33) — (38).

REMARKS

The effect of heat addition seems to be prominent as appears from the expression of the strength of the shock. If once this strength has a perceptible change, all the state variables and their differentials also undergo corresponding changes. In the absence of facilities for numerical calculations, comparison with the case of ordinary fluids was not possible.

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