# STOCHASTIC DUEL WITH SEVERAL TYPES OF WEAPONS

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(*Received* 29 *April* 66; *revised* 23 *Aug.* 66)

This paper attempts to study a 'Stochastic Duel' model wherein each of the contestants has got more than one type of weapons. The ultimate probability of win for each of them is evaluated and the results for a few particular cases are given.

Ancker & Williams[1] have considered a combat situation by means of a theoretical stochastic model named as 'Stochastic Duel'. The model finds immediate interesting applications in analysing problems of combat. The analysis of the model takes into account the details of the situation such as the firing rate of contestants, the single shot kill probability of the rounds, the stock position of the rounds with them and many other related aspects of cover, concealment, surprise etc.

The method of analysis followed by Ancker[2,3] is based on probability arguments He finds the time taken by a duelist to secure a kill. The duelist who takes less time to kill than his opponent ultimately wins the duel. Utilising the convolution property of characteristic function, Ancker obtains the modified Fourier transform of the time to kill the opponent from which is derived the probability of a win as a contour integral, which can be evaluated by the usual method.

In the various models treated so far, it has been assumed that each of the duelists has only one kind of weapon. In actual combat situation this may not be the case especially when we take into consideration the different types of weapons that are used in modren warfare. Further, we may regard the duelist not as a single combatant but as composed of a group of combatants using different kinds of weapons. In such situations the firing rate and the kill probability per round fired will differ from weapon to weapon. To explain the nature of the problem that arises in these circumstances, let us suppose that the duelist has a weapon which requires longer time to fire but its kill probability is much higher than his other weapon which takes a comparatively shorter time to fire. This poses an important operational research problem of determining the pattern of deployment of different weapons with varying kill power. The decision will be all the more important and to a certain extent complicated if the opponent too is in possession of different types of weapons and it is not known how he deploys them.

The above considerations have stimulated the author to consider a duel model wherein each of the duelist possesses several kinds of weapons. As the weapons are not identical, their firing rates and single shot kill probabilities have been taken to be different. It is felt that this kind of investigation may be quite useful to the commander of a task force in deploying his weaponry. Besides, this may also help in providing an assessment of one's own winning capacity when faced with an enemy of known kill power.

The method of analysis in this paper consists in setting up a system of difference—differential equations describing the process which has been solved by the technique of generating functions. This kind of analysis has been widely used in studying problems of stochastic nature particularly encountered in birth-death processes and queues.

## STATEMENT OF THE PROBLEM

Two duelists designated as $A$ and $B$ are engaged in a duel with the aim of securing a kill over the opponent. Duelist $A$ has got k kinds of weapons with different firing rates. It is assumed that the firing times of the $A$'s *i-th* weapon follows exponential distribution with parameter $\lambda_i$ $(i = 1,\ldots\ldots\ldots k)$. The single shot kill probability of each round fired from the *i-th* type of weapon is $p_i$ $(i = 1,\ldots\ldots\ldots k)$. The survival probability of $B$ after $A$ has fired a round from his *i-th* weapon is $q_i$ $(i = 1\ldots\ldots k)$ where $p_i+q_i = 1$. Parameters $\lambda'_i$ and the probabilities $p'_j$ and $q'_j$ in respect of the $r$ weapons used by $B$ are also similarly defined.

It has been assumed here that when a duelist is able to kill his opponent, the firing process terminates and the duel is decided in his favour. It is also assumed that the duelists have an infinite number of rounds at their disposal for all kinds of weapons. Initially they start the duel with no firing from either side.

## FORMULATION OF THE PROBLEM

To formulate the problem thus posed, we proceed to define the following probabilities :

$P(n_1,n_2,\ldots\ldots\ldots n_k : m_1, m_2, \ldots\ldots\ldots m_r ; t)$—The probability that at time $t$ the firing process is in the stage $(n_1,n_2,\ldots\ldots n_k ; m_1, m_2,\ldots\ldots m_r)$ i.e. $A$ has fired $n_i$ rounds of the *i-th* kind $(i = 1,\ldots\ldots k)$ while $B$ has fired $m_j$ rounds of the *j-th* kind $(j=1,\ldots\ldots r)$ and both of them are still alive.

$Q_i(n_1,n_2,\ldots\ldots\ldots n_k : m_1, m_2,\ldots\ldots m_r ; t)$ $(i=1,2,\ldots\ldots\ldots k)$—The probability that at time $t$ the firing process is in the stage $(n_1,n_2,\ldots\ldots\ldots n_k : m_1,\ldots\ldots\ldots m_r)$ and the $n_i$*-th* round fired by $A$ from his *i-th* kind of weapon has killed $B$.

$S_j$ $(n_1, n_2,\ldots\ldots\ldots n_k : m_1, m_2,\ldots\ldots\ldots m_r ; t)$ $(j=1,2,\ldots\ldots\ldots r)$—The probability that at time $t$ the firing process in the stage $(n_1,n_2,\ldots\ldots\ldots n_k ; m_1, m_2,\ldots\ldots\ldots m_k)$ and the $m_j$*-th* round fired by $B$ from *j-th* kind of weapon has killed $A$.

By connecting the state of the system at time $t$ with that at $(t+\triangle)$ we get the following equations governing the process:

$$P(n_1,\ldots n_k ; m_1, \ldots m_r ; t+\triangle) = \prod_{i=1}^{k}(1-\lambda_i \triangle) \prod_{j=1}^{r}(1-\lambda'_j \triangle) P(n_1,\ldots\ldots\ldots n_k ; m_1, m_2,\ldots\ldots\ldots m_r, t)$$
$$+ \sum_{i=1}^{k} \lambda_i q_i P(n_1,\ldots n_{i}-_1,\ldots n_k ; m_1, \ldots m_r, t)$$
$$+ \sum_{j=1}^{r} \lambda'_j q'_j P(n_1,..n_k ; m_1..m_j-_1,..m_r ; t) \quad \text{for } n_1\ldots n_k ; m_1\ldots m_r \geqslant 1. \tag{1}$$

$$P(n_1, n_2,\ldots n_k ; m_1,\ldots m_r ; t+\triangle) = \prod_{i=1}^{k}(1-\lambda_i \triangle) \prod_{j=1}^{r}(1-\lambda'_j \triangle) P(n_1, \ldots n_k, m_1\ldots m_r, t)$$
$$\text{for } n_1 = n_2\ldots = n_k = m_1 = \ldots = m_r = 0$$

$$Q_i\ (n_1, n_2, \ldots n_k\ ; m_1, \ldots m_r\ ; t + \triangle) = Q_i\ (n_1, \ldots n_k\ ; m_1, \ldots m_r, t\ ) + p_i\ \lambda_i\ \ P\ (n_1, \ldots n_{i-1}, \ldots n_k\ ;\ m_1, \ldots m_r,\ t)\ \text{for}\ n_i \geqslant 1 \tag{2}$$

$$Q_i\ (n_1, \ldots n_k\ ;\ m_1, \ldots m_r\ , t\ +\ \triangle) = Q_i\ (n_1, \ldots n_k\ ;\ m_1, \ldots m_r, t)\quad \text{for}\ n_i = 0\ (i = 1, \ldots k)$$

$$S_j\ (n_1, \ldots n_k\ ; m_1, \ldots m_2, t + \triangle) = S_j\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ , t) + \lambda'_j\ p'_j\ P\ (n_1, \ldots n_k\ ; m_1, \ldots m_{j-1} \ldots m_r, t)\quad \text{for}\ m_j \geqslant 1 \tag{3}$$

$$S_j\ (n_1, \ldots n_k\ ;\ m_1, \ldots m_r\ , t\ +\ \triangle) = S_j\ (n_1, \ldots n_k\ ;\ m_1, \ldots m_r\ , t)\quad \text{for}\ m_j = 0\ (j = 1, 2, \ldots r)$$

By rearranging and taking the limit as $\triangle \to 0$ we get the following difference differential equation:

$$\left(\frac{d}{dt} + \sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{r} \lambda'_j\right) P\ (n_1, \ldots n_k\ ; m_1 \ldots m_r, t) = \sum_{i=1}^{k} \lambda_i\ q_i\ P\ (n_1, \ldots n_{i-1}, \ldots n_k\ ; m_1, \ldots m_r,\ t) + \sum_{j=1}^{r} \lambda'_j\ q'_j\ P\ (n_1, \ldots n_k\ ; m_1, \ldots m_{j-1}, \ldots m_r\ ,t) \tag{4}$$

$$\text{for}\ n_1, \ldots n_k\ ;\ m_1, \ldots m_r \geqslant 1$$

$$\left(\frac{d}{dt} + \sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{r} \lambda'_j\right) P\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ ;\ t) = 0$$

$$\text{for}\ n_1 = \ldots = n_k = \ldots = m_1 = m_r = 0$$

$$\frac{d}{dt}\ Q_i\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ , t) = \lambda_i p_i\ \ P\ (n_1, \ldots n_{i-1}, \ldots n_k\ ;\ m_1, \ldots m_r\ , t)\quad \text{for}\ n_i \geqslant 1$$

$$= 0 \quad \text{for}\ n_i = 0 \tag{5}$$

$$\frac{d}{dt} S_j\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ , t) = \lambda'_j\ p'_j\ P\ (n_1, \ldots n_k\ ;\ m_1 \ldots m_{i-1}, \ldots m_r\ , t)\quad \text{for}\ m_j \geqslant 1$$

$$= 0 \quad \text{for}\ m_j = 0 \tag{6}$$

$$\text{Initially}\ P\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ , 0) = \prod_{i=1}^{k} \delta_{n_i,\,0} \prod_{j=1}^{r} \delta_{m_j,\,0} \tag{7}$$

where $\delta_{ij}$ is the Kronecker's dalta.

To solve the above equation we define the following generating function:

$$F\ (\ (\alpha\ )\ ;\ (\ \beta\ ), t\ ) = \sum_{n_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left[ \prod_{i=1}^{k} \alpha_i^{n_i} \prod_{j=1}^{r} \beta_j^{m_j} \right] P\ (n_1, \ldots n_k\ ; m_1, \ldots m_r\ ;\ t)$$

where $(\alpha)$ stands for $(\alpha_1, \alpha_2 \ldots\ldots \alpha_k)$ and $\beta$ stands for $(\beta_1, \beta_2, \ldots\ldots \beta_r)$.

$$A_i\ ((\alpha); (\beta); t) = \sum_{n_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left[ \prod_{i=1}^{k} \alpha_i^{n_i} \prod_{j=1}^{r} \beta_j^{m_j} \right] Q_i(n_1, \ldots n_k; m_1, \ldots m_r, t)$$

$$B_j\ ((\alpha); (\beta); t) = \sum_{n_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left[ \prod_{i=1}^{k} \alpha_i^{n_i} \prod_{j=1}^{r} \beta_j^{m_j} \right] S_j\ (n_1, \ldots n_k; m_1, \ldots m_r, t)$$

Applying the above defined generating function to the equations (4) to (7) we get

$$\left( \frac{d}{dt} + \sum_{i=1}^{k} \lambda_i\ (1 - q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_j) \right) F\ ((\alpha); (\beta); t) = 0 \qquad (8)$$

$$\frac{d}{dt}\ A_i\ ((\alpha); (\beta); t) = \lambda_i\ p_i\ \alpha_i\ F\ ((\alpha); (\beta); t) = \psi_i\ ((\alpha); (\beta); t)\ \text{(say)} \qquad (9)$$

$$\frac{d}{dt}\ B_j\ ((\alpha); (\beta); t) = \lambda'_j\ p'_j\ \beta_j\ F\ ((\alpha); (\beta); t) = \phi_j\ ((\alpha); (\beta); t)\ \text{(say)} \qquad (10)$$

with the initial condition

$$F\ ((\alpha); (\beta); 0) = 1 \qquad (11)$$

Solving (8) with the initial condition (11) we get

$$F\ ((\alpha); (\beta); t) = \exp \left[ - \left\{ \sum_{i=1}^{k} \lambda_i\ (1 - q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_j) \right\} \right] t \qquad (12)$$

Substituting the value of $F[(\alpha); (\beta); t]$ in (9) and (10) we obtain the generating function of the density of kill for the duelists as

$$\psi_i\ ((\alpha); (\beta); t) = \lambda_i\ p_i\ \alpha_i\ \exp \left[ - \left\{ \sum_{i=1}^{k} \lambda_i\ (1 - q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_j) \right\} t \right] \qquad (13)$$

$$\varphi_j\ ((\alpha); (\beta); t) = \lambda'_j\ p'_j\ \beta_j\ \exp \left[ - \left\{ \sum_{i=1}^{k} \lambda_i (1 - q_i \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_j) \right\} t \right] \qquad (14)$$

Integrating with respect to the time variable $t$ we obtain the generating functions of the kill probability for the duelist $A$ when the kill is secured over the duelist $B$ by the *i-th* weapon of $A$ as :

$$\psi_i\ ((\alpha); (\beta)) = \lambda_i\ p_i\ \alpha_i\ / \left[ \sum_{i=1}^{k} \lambda_i\ (1 - q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_j) \right] \qquad (15)$$

Similarly the generating function $\phi_j\ [(\alpha); (\beta)]$ for the kill probability that the duelist $B$ kills his opponent $A$ by the *j-th* weapon is given by

$$\varphi_j\ ((\alpha); (\beta)) = \lambda'_j\ p'_j\ \beta_i / \left[ \sum_{i=1}^{k} \lambda_i\ (1 - q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda'_j\ (1 - q'_j\ \beta_i) \right] \qquad (16)$$

These give the generating functions, $P_{k,r}$ $[A; (\alpha); (\beta)]$ and $P_{k,r}$ $[B; (\alpha); (B)]$ of the probability of ultimate win for the duelist $A$ and $B$ with $A$ deploying $k$ types of weapons and $B$ deploying $r$ types of weapons as

$$P_{k,r}(A; (\alpha); (\beta)) = \sum_{i=1}^{k} \lambda_i(1-q_i\ \alpha_i) / [\sum_{i=1}^{k} \lambda_i\ (1-q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda_j'\ (1-q_j'\ \beta_j\ )] \tag{17}$$

$$P_{k,r}(B; (\alpha); (\beta)) = \sum_{j=1}^{r} \lambda_j'\ (1-q_j'\ \beta_j) / [\sum_{i=1}^{k} \lambda_i\ (1-q_i\ \alpha_i) + \sum_{j=1}^{r} \lambda_j'\ (1-q_j'\ \beta_j)] \tag{18}$$

Putting $\alpha_i = \beta_j = 1\ (i = 1, \ldots\ldots k; j = 1 \ldots\ldots r)$ in (17) we obtain $P_{k,r}(A)$ the probability that duelist $A$ wins the duel when he deploys $k$ types of weapons and his opponent deploys $r$ types of weapons.

$$P_{k,r}(A) = \sum_{i=1}^{k} \lambda_i\ p_i\ / [\sum_{i=1}^{k} \lambda_i\ p_i + \sum_{j=1}^{r} \lambda_j'\ p_j'\ ] \tag{19}$$

Similarly we obtain $P_{k,r}(B)$ the probability that the duelist $B$ deploying $r$ types of weapons wins the duel over his enemy deploying $k$ types of weapons.

$$P_{k,r}(B) = \sum_{j=1}^{r} \lambda_j'\ p_j'\ / [\sum_{i=1}^{k} \lambda_i\ p_i + \sum_{j=1}^{r} \lambda_j'\ p_j'\ ] \tag{20}$$

From (19) we obtain the particular case when both the duelists deploying only one weapon each by putting $k = r = 1$ as

$$P_{1,1}(A) = \lambda_1\ p_1 / [\ \lambda_1\ p_1 + \lambda_1'\ p_1'\ ]$$

which agrees with Ancker's result (ref. 1, eqn. 6)

*Evaluation of some parameters*

We now proceed to calculate the mean and the variance of the rounds spent by the duelists to win the duel. We know that $E(X_i)$—expected number of rounds of the *i-th* kind fired by the duelist $A$ to score a win, is given by

$$E(X_i) = \left[ \frac{\partial}{\partial \alpha_i}\ \psi_i\ \ [\ (\alpha); (\beta)\ ] \right]_{\alpha_i\ =\ \beta_j\ =\ 1}$$

and the variance $V(X_i)$ is given by

$$V(X_i) = \left[ \frac{\partial^2}{\partial \alpha_i^2}\ \psi_i\ ((\alpha); (\beta)) - \frac{\partial}{\partial \alpha_i}\ \Psi_i\ ((\alpha); (\beta)) - \left\{ \frac{\partial}{\partial \alpha_i}\ \psi_i\ ((\alpha); (\beta)) \right\}^2 \right]_{\alpha_i\ =\ \beta_j\ =\ 1}$$

so that

$$E(X_i) = \frac{\lambda_i\ p_i \left[ \lambda_i + \sum\limits_{\substack{n=1 \\ n \neq i}}^{k} \lambda_n\ p_n + \sum\limits_{j=1}^{r} \lambda_j'\ p_j' \right]}{\left[ \sum\limits_{i=1}^{k} \lambda_i\ p_i + \sum\limits_{j=1}^{r} \lambda_j'\ p_j' \right]^2}$$

and

$$\mathrm{Var}\,(X_i) = \frac{2\,\lambda_i^2\, p_i\, q_i \left[ \lambda_i + \sum\limits_{\substack{n=1\\ n\neq i}}^{k} \lambda_n\, p_n + \sum\limits_{j=1}^{r} \lambda_j'\, p_j' \right]}{\left[ \sum\limits_{i=1}^{k} \lambda_i\, p_i + \sum\limits_{j=1}^{r} \lambda_j'\, p_j' \right]^3}$$

Similarly the expectation and variance $E(X'_j)$ and $V(X'_j)$ for the duelist $B$ are given by

$$E\,(X'_j) = \frac{\lambda_j'\, p_j' \left[ \sum\limits_{i=1}^{k} \lambda_i\, p_i + \lambda_j' + \sum\limits_{\substack{n=1\\ n\neq j}}^{r} \lambda_n'\, p_n' \right]}{\left[ \sum\limits_{i=1}^{k} \lambda_i\, p_i + \sum\limits_{j=1}^{r} \lambda_j'\, p_j' \right]^2}$$

and

$$V\,(X'_j) = \frac{2\,\lambda_j'^2\, p_j'\, q_j' \left[ \sum\limits_{i=1}^{k} \lambda_i\, p_i + \lambda_j' + \sum\limits_{\substack{n=1\\ n\neq j}}^{r} \lambda_n'\, p_n' \right]}{\left[ \sum\limits_{i=1}^{k} \lambda_i\, p_i + \sum\limits_{j=1}^{r} \lambda_j'\, p_j' \right]^3}$$

## DISCUSSION

In our analysis we have got factors like $\lambda_i\ p_i$ which is the product of the rate of fire and the single shot kill probability of the *i-th* kind of weapon. This term can safely be interpreted to mean the kill rate of the *i-th* weapon. This we dsignate by $L_i$. If the duelists $A$ and $B$ have got $N_1$ and $N_2$ types of weapons respectively, then the probability of win for the duelist $A$ can be written as

$$P_{N_1, N_2}\ (A) = \sum_{i=1}^{N_1} L_i \Big/ \Big[ \sum_{i=1}^{N_1} L_i + \sum_{j=1}^{N_2} L_j' \Big]$$

where $L'_j$ is the killing rate of the *B's j-th* weapon.

It may be noted here that if $\Sigma\, L_i < \Sigma\, L'_j$ then the probability of win for $A$ is quite high. Reverse is the case when the inequality changes sign. Further if $A$ is overwhelmingly superior to $B$, his chances of winning the duel increase directly as the ratio of his superiority. We find that the outcome of a duel depends upon not the individual weapons killing rate but the sum of killing rates that is available with the duelists from the different weapons.

## ACKNOWLEDGEMENTS

The author is grateful to Dr. N. K. Chakravarti, Assistant Director, and Dr. N.K. Jaiswal, Senior Scientific Officer for guidance and helpful suggestions during the preparation of this paper. Thanks are also due to Dr. Kartar Singh, Director, Defence Science Laboratory for permission to publish this paper.

## REFERENCES

1. WILLIAMS, T. & ANCKER, C.J., *Opns. Res.*, **11** (1963), 803.
2. ANCKER, C.J., *ibid* **12** (1964), 38.
3. ——————— "The Status of Development in the Theory of Stochastic Duels", SP—1017/008/00, (System Development Corporation Santa Monica, California), (1964).