MINIMIZING WAITING TIME OF JOBS OVER TWO MACHINES SCHEDULING PROBLEMS

P.C. BAGGA

Defence Science Laboratory, Delhi

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In certain cases, significant costs are incurred, if there is a waiting time of the jobs. An algorithm which yields an optimum sequence of n jobs, is obtained and is illustrated with an example.

In a machine scheduling problem, the usual procedure is to minimize the waiting time of the last machine[1,2]. In this case interest is attached to the completion of all the jobs in hand, no matter how much time it takes to finish the whole work. As an example, consider the case of two aircraft refuellers needing repairs. Suppose one refueller (X) needs 7 hours for repair which starts at 10 a.m., it will be ready at 5 p.m. If a second refueller (Y) needs one hour for repair, both the refuellers will be ready at 6 p.m., whatever be the sequence of refuellers. But if the total waiting time of these refuellers is considered the sequence XY leads to 7 hours whereas YX yields only one hour. Hence according to this criterion sequence YX is preferable.

In considering the above example, if there are only two refuellers at the air-port and the sequence of repairs is XY, no refuelling is possible up to 6 p.m. since after repairs to X it will have to go to the bulk fuel installation to obtain fuel for delivering the same to an aircraft. If on the other hand, the sequence is YX, Y is repaired by 11 a.m. and will be ready to deliver fuel to an aircraft (if any be waiting for refuelling at that time) by 12 noon. The service to the aircraft is not disrupted throughout the day. The criterion for achieving this sequence is minimization of the total waiting time of the jobs rather than minimisation of the total elapsed time.

In this paper, an algorithm to determine an optimum sequence of n jobs is obtained which when processed on two machines (no passing allowed) yields the minimum total waiting time of the jobs. An example is solved after the algorithm.

PROBLEM

Consider the case of n jobs $J_1, J_2, J_3 \ldots\ldots\ldots J_n$ to be processed on two machines M_1 and M_2, when no passing is allowed. It is further assumed that all the jobs have first to go on to machine M_1. The problem is to find a sequence of these jobs which when processed on these two machines minimizes the total waiting of the jobs—

NOTATIONS

Y_{ij} = processing time taken by ith job on jth machine.

X_{ij} = waiting time of ith job for jth machine.

X_i = total waiting time of ith job before it is completed.

Z_{ij} = time at which ith job is completed on jth machine.

T = total waiting time of jobs.

The problem is to minimize T

where

$$T = \sum_{i=1}^{n} X_i \tag{1}$$

and

$$X_i = X_{i1} + X_{i2}$$

SOLUTION OF PROBLEM

Consider Table 1 which gives completion time of the jobs on these two machines :

TABLE 1

COMPLETION TIME OF JOBS ON MACHINES M_1 & M_2

Jobs	M_1	M_2
J_1	Z_{11}	Z_{12}
J_2	Z_{21}	Z_{22}
J_3	Z_{31}	Z_{32}
⋮	⋮	⋮
J_J	Z_{J1}	Z_{J2}
⋮	⋮	⋮
J_n	Z_{n1}	Z_{n2}

Let us evaluate the values of $X_1, X_2, X_3 \ldots\ldots\ldots\ldots X_n$

Now

$$\begin{aligned} X_1 &= X_{11} + X_{12} \\ &= 0 + 0 \\ X_2 &= Z_{11} + Z_{12} - Z_{21} && \text{if } Z_{12} > Z_{21} \\ &= Z_{11} + 0 && \text{if } Z_{12} \leqslant Z_{21} \\ &= Z_{11} + max\,(Z_{12} - Z_{21},\ 0) \end{aligned}$$

similarly

$$\begin{aligned} X_3 &= Z_{21} + max.\,(Z_{22} - Z_{31},\ 0) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{J-1} &= Z_{J-21} + max.\,(Z_{J-22} - Z_{J-11},\ 0) \\ X_J &= Z_{J-11} + max.\,(Z_{J-12} - Z_{J1},\ 0) \\ X_{J+1} &= Z_{J1} + max.\,(Z_{J2} - Z_{J+11},\ 0) \\ X_{J+2} &= Z_{J+11} + max.\,(Z_{J+12} - Z_{J+21},\ 0) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_n &= Z_{n-11} + max.\,(Z_{n-12} - Z_{n1},\ 0) \end{aligned}$$

where

$$Z_{K1} = \sum_{i=1}^{K} Y_{i1} \tag{3}$$

and

$$Z_{K2} = max\,[\,Z_{K1},\ Z_{K-12}\,] + Y_{J2} \tag{4}$$

Theorem—The determination of the optimum sequence is given by the following rule job J_J will precede job J_{J+1} if

$$Y_{J1} + max.\,[\,Z_{J-12} - \sum_{i=1}^{J} Y_{i1},\, 0\,] + max.\,[\,Y_{J2} + max.\,(\,Z_{J-12},\, \sum_{i=1}^{J} Y_{i1}\,) - \sum_{i=1}^{J+1} Y_{i1},\, 0]$$

$$< Y_{J+11} + max.\,[\,Z_{J-12} - \sum_{i=1}^{J-1} Y_{i1} - Y_{J+11},\, 0\,]$$

$$+ max.\,[\,Y_{J+12} + max.\,(Z_{J-12},\, \sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}\,) - \sum_{i=1}^{J+1} Y_{i1},\, 0] \tag{5}$$

and

$$Y_{J+12} + max.\,[\,\sum_{i=1}^{J+1} Y_{i1} + max.\,(\,Z_{J-12},\, \sum_{i=1}^{J} Y_{i1}\,)]$$

$$\leqslant Y_{J2} + max.\,[\,\sum_{i=1}^{J+1} Y_{i1},\, Y_{J+12} + max.\,(\,\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11},\, Z_{J-12}\,)] \tag{6}$$

which on simplification becomes

$$min.\,[\,Y_{J1},\, Y_{J+12}\,] \leqslant min.\,[\,Y_{J+11},\, Y_{J2}\,] \tag{7}$$

Proof—Let us start with a sequence S and from it we obtain another sequence S' by interchanging the jth and $(j+1)$ th jobs.

The two sequences may now be written as

$S = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots j-1, j, j+1, j+2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots Jn$

$S' = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots j-1, j+1, j, j+2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots Jn$

If T and T' determine the total waiting of the jobs in these two sequences respectively, then sequence S will be preferable over S' if

$$T < T' \tag{8}$$

If X'_i denotes the corresponding values for sequence S' then

$X_k = X'_k$ $\qquad k = 1, 2, 3, \ldots\ldots\ldots\ldots j-1.$

but $X_J, X_{J+1}, X_{J+2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots X_n$ may not be same to its corresponding values in sequence S'.

Now evaluate the values of X's for sequences S and S'

For Sequence S (from 2)

$$X_J = Z_{J-11} + max\,(\,Z_{J-12} - Z_{J1},\, 0\,)$$

substituting the values of Z_{J-11}, Z_{J1} from (3), one gets

$$= \sum_{i=1}^{J-1} Y_{i1} + max\,(\,Z_{J-12} - \sum_{i=1}^{J} Y_{i1},\, 0) \tag{9}$$

$$X_{J+1} = Z_{J1} + max. [\ Z_{J2} - Z_{J+11},\ 0\ ]$$

$$= \sum_{i=1}^{J} Y_{i1} + max. [\ Y_{J2} + max. (\ Z_{J1},\ Z_{J-12}\ ) - Z_{J+11},\ 0\ ]$$

$$= \sum_{i=1}^{J} Y_{i1} + max. [\ Y_{J2} + max. (\ Z_{J1},\ Z_{J-12}\ ) - Z_{J+11},\ 0\ ] \qquad (10)$$

Similarly

$$X_{J+2} = \sum_{i=1}^{J+1} Y_{i1} + max. [\ ((\ Y_{J+12} + max. \{ \sum_{i=1}^{J+1} Y_{i1},\ Y_{J2} + max. (\sum_{i=1}^{J} Y_{i1},\ Z_{J-12}) \})) - \sum_{i=1}^{J+2} Y_{i1},\ 0\ ] \qquad (11)$$

$$X_{J+3} = \sum_{i=1}^{J+2} Y_{i1} + max. \Big[\ Y_{J+22} + max. \{ \sum_{i=1}^{J+2} Y_{i1},\ ((\ Y_{J+12} + max. [\sum_{i=1}^{J+1} Y_{i1},\ Y_{J2} + max. (\sum_{i=1}^{J} Y_{i1},\ Z_{J-12}\ )])) \} - \sum_{i=1}^{J+3} Y_{i1},\ 0\ \Big] \qquad (12)$$

....

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$$X_n = \sum_{i=1}^{n-1} Y_{i1} + max. \Big[\ Y_{n-12} + max. \{ \sum_{i=1}^{n-1} Y_{i1},\ Y_{n-22} + \ldots\ldots \ldots\ldots Y_{J+22} + max. [\sum_{i=1}^{J+2} Y_{i1},\ ((\ Y_{J+12} + max. \{ \sum_{i=1}^{J+1} Y_{i1},\ Y_{J2} + max. (\sum_{i=1}^{J} Y_{i1},\ Z_{J-12}) \}))] \ldots \} - \sum_{i=1}^{n} Y_{i1},\ 0\ \Big] \qquad (13)$$

The corresponding values for sequence S' are

$$X'_J = \sum_{i=1}^{J-1} Y_{i1} + max. [\ Z_{J-12} - \sum_{i=1}^{J-1} Y_{i1} - Y_{J+11},\ 0\ ] \qquad (14)$$

$$X'_{J+1} = \sum_{i=1}^{J-1} Y_{i1} + Y_{J+11} + max. [\ Y_{J+12} + max. (Z_{J-12}, \sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}) - \sum_{i=1}^{J+1} Y_{i1},\ 0\ ] \qquad (15)$$

$$X'_{J+2} = \sum_{i=1}^{J+1} Y_{i1} + max. [\ ((\ Y_{J2} + max. \{ \sum_{i=1}^{J+1} Y_{i1},\ Y_{J+12} + max. (\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11},\ Z_{J-12}\ ) \})) - \sum_{i=1}^{J+2} Y_{i1},\ 0\ ] \qquad (16)$$

$$X'_{J+3} = \sum_{i=1}^{J+2} Y_{i1} + max. \Big[Y_{J+22} + max. \; \{ \sum_{i=1}^{J+2} Y_{i1}, \; ((Y_{J2} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J+12} + max. (\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}, Z_{J-12})])) \} - \sum_{i=1}^{J+3} Y_{i1}, \; 0 \Big] \tag{17}$$

....

....

....

$$X'_n = \sum_{i=1}^{n-1} Y_{i1} + max. \Big[Y_{n-12} + max. \; \{ \sum_{i=1}^{n-1} Y_{i1}, \; Y_{n-22} + \ldots\ldots + max. [\sum_{i=1}^{J+2} Y_{i1}, \; ((Y_{J2} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J+12} + max. (\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}, Z_{J-12})]))]..\} - \sum_{i=1}^{n} Y_{i1}, \; 0 \Big] \tag{18}$$

From the above expressions it can be easily seen that X_k is the same as the corresponding X'_k ($k \geqslant J+2$) except for the expressions as shown in the ((..)).

Hence

$$X_{J+2} \leqslant X'_{J+2}, \; X_{J+3} \leqslant X'_{J+3}, \ldots\ldots\ldots\ldots X_n \leqslant X'_n \tag{19}$$

if

$$Y_{J+12} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J2} + max. (\sum_{i=1}^{J} Y_{i1}, \; Z_{J-12})]$$
$$\leqslant Y_{J2} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J+12} + max. (\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}, \; Z_{J-12})]$$

i.e

$$Y_{J+12} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J2} + \sum_{i=1}^{J} Y_{i1}, \; Z_{J-12} + Y_{J2}]$$
$$\leqslant Y_{J2} + max. [\sum_{i=1}^{J+1} Y_{i1}, \; Y_{J+12} + max. (\sum_{i=1}^{J-1} Y_{i1} + Y_{J+11}, Z_{J-12})]$$

or

$$max. [\sum_{i=1}^{J+1} Y_{i1} + Y_{J+12}, \sum_{i=1}^{J} Y_{i1} + Y_{J2} + Y_{J+12}, Z_{J-12}, + Y_{J2} + Y_{J+12}]$$
$$\leqslant max. [\sum_{i=1}^{J+1} Y_{i1} + Y_{J2}, \sum_{i=1}^{J-1} Y_{i1} + Y_{J+11} + Y_{J2} + Y_{J+12}, Z_{J-12} + Y_{J2} + Y_{J+12}]$$

which will be always true if

$$max. \left[\sum_{i=1}^{J+1} Y_{i1} + Y_{J+12},\ \sum_{i=1}^{J} Y_{i1} + Y_{J2} + Y_{J+12} \right]$$

$$\leqslant max. \left[\sum_{i=1}^{J+1} Y_{i1} + Y_{J2},\ \sum_{i=1}^{J-1} Y_{i1} + Y_{J+11} + Y_{J2} + Y_{J+12} \right]$$

Substracting

$$\sum_{i=1}^{J+1} Y_{i1} + Y_{J2} + Y_{J+12}$$

we get,

$$max. \left[-Y_{J2},\ -Y_{J+11} \right] \leqslant max \left[-Y_{J+12},\ -Y_{J1} \right]$$

$$i\,e\ min \left[Y_{J+11},\ Y_{J2} \right] \geqslant min \left[Y_{J+12},\ Y_{J1} \right]$$

Thus (19) in a simplified form becomes

$$min. \left[Y_{J1},\ Y_{J+12} \right] \geqslant min \left[Y_{J+11},\ Y_{J2} \right] \qquad (20)$$

which is also the required condition to find an optimal sequence in case of two machines given by Johnson.

Thus job J_J should precede job J_{J+1} if

$$X_J + X_{J+1} < X'_J + X'_{J+1}$$

and

$$min. \left[Y_{J1},\ Y_{J+12} \right] \leqslant min. \left[Y_{J+11},\ Y_{J2} \right]$$

Substituting the values of X_J, X_{J+1}, X'_J, X'_{J+1}

and simplifying we get

$$Y_{J1} + max. \left[Z_{J-12} - \sum_{i=1}^{J} Y_{i1}, 0 \right] + max. \left[Y_{J2} + max. \left(\sum_{i=1}^{J} Y_{i1}, Z_{J-12} \right) - \sum_{i=1}^{J+1} Y_{i1}, 0 \right]$$

TABLE 2

SCHEDULED SEQUENCE POSITIONS OF (j-1) JOBS

Sequence Position	Job i	Yi_1	Yi_2
1	6	Y_{61}	Y_{62}
2	9	Y_{91}	Y_{92}
3	1	Y_{11}	Y_{12}
:	:	:	:
:	:	:	:
:	:	:	:
:	:	:	:
J-1	4	Y_{41}	Y_{42}
:	:	:	:
:	:	:	:
:	:	:	:
:	:	:	:
:	:	:	:
:	:	:	:
n	:	:	:

$$<Y_{J+11}+max.\,[Z_{J-12}-\sum_{i=1}^{J-1} Y_{i1}-Y_{J+11},0]$$

$$+max.\,[Y_{J+12}+max.\,(\sum_{i=1}^{J-1} Y_{i1}+Y_{J+11},Z_{J-12})-\sum_{i=1}^{J+1} Y_{i1},0] \tag{21}$$

(20) and (21) are the required conditicns.

ALGORITHM

Let us assume that $(j-1)$ jobs have been scheduled for the fesible sequence under consideration. The procedure for the determination of the *jth* position is as follows—

Step 1—List the $(j-1)$ scheduled jobs in their scheduled sequence positions as illustrated in Table 2.

Step 2—Determine the *min.* Y_{i1} for all the remaining unscheduled jobs. If there is a tie, select the one with the *max* Y_{i2}. Place the corresponding job and its processing times in the *j*th sequence position of the sequence Table.

Step 3—Place one of the remaining $(n-j)$ unscheduled jobs in the $(j+1)$ st sequence position of the sequence Table. For the sake of definitness and to ensure that no possible sequence is overlooked, select the job with the smallest subscript first.

Step 4—Determine if equation (5) is satisfied.

Step 5—Apply one of the following:

(*a*) If equation (5) is satisfied, repeat steps 3 to 5 for each remaining possible sequence i.e. the reduced set of (n-1-j), (n-2-j), etc., remaining unscheduled jobs : continue to step *6*.

(*b*) If equation (5) is not satisfied because of an equality, repeat steps *3* through *5* for each remaining possible sequence : continue to Step *6*.

(*c*) If equation (5) is otherwise not satisfied replace the job currently in the sequence position (j) with the job currently in the sequence $(j+1)$, repeat steps *3* through *5* for each remaining possible sequence, if all unscheduled jobs have been tried in the sequence position (j) proceed to step *6*.

Step 6—One of the following will be apparent after the completion of the preceding steps, apply the appropriate condition:

(*a*) The job presently in the jth position satisfies equ. (5) for all remaining unscheduled jobs, go to step *7a*.

(*b*) The job presently in the *j*th position fails to satisfy equ. (5) because one or more of the remaining unscheduled jobs yield an equality, go to step *7b*.

(*c*) None of the remaining unscheduled jobs satisfied (*a*) or (*b*) above, go to step *7c*.

Step 7—Apply one of the following:

(*a*) If *6a* occurs, determine if equation (6) is satisfied for all the remaining $(n-j)$ unscheduled jobs.

(1) If equation (6) is satisfied for all remaining unscheduled jobs, schedule the job in the sequence position (j) as the next job of the feasible sequence.

(2) If equation (6) is not satisfied for i remaining $(n-j)$ unscheduled jobs, it is necessary to assume that the job presently in the jth position as well as the remaining i unscheduled jobs that do not satisfy this condition as the jth job of $(i+1)$ feasible sequences.

(*b*) If *6b* occurs, it is necessary to assume the job in the jth sequence position as well as k of the remaining unscheduled jobs as the jth job of $(k+1)$ feasible sequence, where k is equal to the number of remaining unscheduled jobs that had yielded an equality in the test of equation (5).

(*c*) If *6c* occurs, it is necessary to assume all remaining $(n+1-j)$ unscheduled jobs as the j*th* job of $(n+1-j)$ feasible sequences.

Step 8—Having found or assumed one job to be placed in the jth position of the sequence Table, repeat Steps *1* through *7* until $(n-2)$ jobs have been sequenced into a feasible solution. (If more than one job were assumed in *Step 7*, the first is assigned to position j and others are put aside until *Step 10*).

Step 9—Enumerate the feasible sequence to determine total waiting time.

Step 10—If more than one job were assumed for position j in *step 7* select another (the first was choseen in *step 8*) and repeat *step 1* through *9* until all values put aside in step 8 are used.

TABLE 3

PROCESSING TIMES OF SIX JOBS ON MACHINE M_1 & M_2.

Job	M_1	M_2
1	3	8
2	12	4
3	5	9
4	2	6
5	9	7
6	11	1

TABLE 4

SEQUENCE POSITION OF JOB 4 *vs.* JOB 1

Sequence Position	Job	M_1	M_2
1	4	2	6
2	1	3	8

Step 11—Determine the sequence (one or more) that yields the minimum waiting time of the jobs under consideration. This sequence is the required sequence as it minimizes the total waiting time T.

EXAMPLE

Let us have six jobs each of which has to go through two machines M_1 and M_2 in order $M_1 M_2$. The processing times are given in Table 3

The problem is to determine a sequence of six jobs which minimizes the total waiting time.

Solution

As the min. Y_{i1} is 2 hrs. and that is for job 4, hence it is placed in the sequence position 1 of the sequence Table 4. We now place job 1 (say) at the sequence posititon 2 and verify equation (5).

$$2 + max. \, [\, 0 - 2, \, 0 \,] + max. \, [\, 6 + max. \, (0, \, 2) - 5, \, 0 \,]$$
$$< 3 + max. \, [\, 0 - 3, \, 0 \,] + max. \, [\, 8 + max. \, (\, 0, \, 3 \,) - 5, \, 0 \,]$$

where

$$Z_{01} = 0, \quad Z_{02} = 0, \quad Y_{J1} = 2, \quad Y_{J2} = 6, \quad Y_{J+11} = 3 \ \text{ and } \ Y_{J+12} = 8$$

Since equation (5) is satisfied, we now verify the condition against job 2,3,5,6. It can be very easily seen that all the jobs when placed at the second position satisfy equation (5). As all the jobs satisfy equation (5), we now verify (6).

Similarly it can be seen that all the jobs satisfy condition 2. Hence job 4 takes the I position of the feasible sequence.

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After deciding the first position, we again look for *min.* Y_{i1} ($i \neq 4$). Job takes the second position and we verify (5) and (6) against jobs 2,3,5,6. It can be seen that all the jobs satisfy and thus job 1 takes the second position of the feasible sequence.

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Having scheduled job 4 and job 1 at the sequence positions 1 and 2 respectively, we place job 3 at the third position and verify equation (5) against job 2.

Job 3 vs. Job 2

$$5 + max. \, [\, 16 - 10, \, 0 \,] + max. \, [\, 9 + max. \, (\, 16, \, 10 \,) - 22, \, 0 \,]$$
$$< 12 + max. \, [\, 16 - 17, \, 0 \,] + max. \, [\, 4 + max. \, (\, 16, \, 17 \,) - 22, \, 0 \,]$$

TABLE 5

(SEQUENCE A)

Job	Machine M_1		X_{i1}	Machine M_2		X_{i2}	X_i
	In	Out		In	Out		
1	2	3	4	5	6	7	8
4	0	2	0	2	8	0	0
1	2	5	2	8	16	3	5
5	5	14	5	16	23	2	7
3	14	19	14	23	32	4	18
2	19	31	19	32	36	1	20
6	31	42	31	42	43	0	31

$\Sigma X_i = 81$ hrs.

TABLE 6*

(SEQUENCE B)

1	2	3	4	5	6	7	8
4	0	2	0	2	8	0	0
1	2	5	2	8	16	3	5
5	5	14	5	16	23	2	7
3	14	19	14	23	32	4	18
6	19	30	19	32	33	2	21
2	30	42	30	42	46	0	30

$\Sigma X_i = 81$ hrs

TABLE 7*

(SEQUENCE C)

1	2	3	4	5	6	7	8
4	0	2	0	2	8	0	0
1	2	5	2	8	16	3	5
6	5	16	5	16	17	0	5
3	16	21	16	21	30	0	16
5	21	30	21	30	37	0	21
2	30	42	30	42	46	0	30

$\Sigma X_i = 77$ hrs.

*Columns 1—8 are the same as in TABLE 5.

Job 3 *vs. Job* 5

$$5 + max. [\, 16 - 10,\ 0\,] + max. [\, 9 + max. (\, 16,\ 10\,) - 19,\ 0\,]$$
$$< 9 + max. [\, 16 - 14,\ 0\,) + max. [\, 7 + max. (\, 16,\ 14\,) - 19,\ 0\,]$$

Here (5) is not satisfied, therefore we remove job 3 from sequence position 3 and replace it by job 5.

Job 5 *vs. Job* 2

$$9 + max. [\, 16 - 14,\ 0\,] + max. [\, 7 + max. (\, 16,\ 14\,) - 26,\ 0\,]$$
$$< 12 + max. [\, 16 - 17,\ 0\,] + max. [\, 4 + max. (\, 16,\ 17\,) - 26,\ 0\,]$$

Job 5 *vs Job* 6

$$9 + max. [\,16-14,0\,] + max. [\,7 + max. (\,16,14\,) - 25,0\,]$$
$$< 11 + max. [\,16-16,\ 0\,] + max. [\,1 + max. (\,16,14\,) - 25,0\,]$$

Since an equality exists (11=11), it is necessary to assume both job 5 and job 6 as the third job of the two feasible sequences.

SEQ.A=415

SEQ.B=416

continuing in this way, we finally get three feasible sequences.

A= 415326
B= 415362
C= 416352

To decide, which of these sequences is the optimum one, we enumerate these sequences separately. The details are given in Tables 5, 6, and 7.

Since sequence C gives the minimum waiting time, it therefore becomes the optimum sequence.

DISCUSSION

Johnson[1] and Teuton[2] have been interested in minimizing the cost of operating the machines. Here the installation and operational costs of machines are heavy and hence minimisation over that last machine results in their best utilization. They have considered that the waiting time of the jobs is not as important as that of machines. This may or may not be so in all the cases as have been illustrated with an example. Moreover by trying to minimize the waiting time of jobs we are led to the conditions, one of which is same as given by Johnson. Hence our solution of the problem automatically results in minimization over the last machine as well as the minimization of the waiting time of the jobs. We have thus arrived at a result that is more general than the one proposed by Johnson. It is now quite clear that our criterion will include at least one sequence that will also minimize the waiting time over the last machine. In the example discussed above, we find that the sequence A (415326) results in minimum total elapsed time which is 43 hours and the waiting time of the jobs amounts to 81 hrs. According to Johnson, the corresponding figures are 43 hr. and 84 hr. respectively. Thus even Johnson's criterion leads to same waiting time of the last machine, it does not obtain the sequence A which over above minimizing the waiting of the last machine also results in less waiting of jobs. Our approach, therefore leads to few sequences, some of which will be at least as good as given by Johnson, if not better.

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REFERENCES

1. JOHNSON, S. M., *Naval Res. Log. Quart.*, 1 (1954), 61
2. DUDEK, R. A. & TEUTON, O.E., *Opn. Res.*, 12 (1964), 471