ON THE CALCULATION OF OPTIMUM MASS DISTRIBUTION OF A MULTI-STAGE ROCKET VEHICLE

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The effect of gravity on the optimum distribution of total required mass among the various stages of a multiple stage rocket arranged in series has been considered by making the payload ratio minimum so as to obtain a specified mission velocity at the end of powered flight. The special case when the physical parameters for all the step-rockets are each equal, has been discussed in detail. It has also been shown that if the mission requirement is to achieve a given all-burnt height, then even at the expense of more total initial mass no more total height is obtained than in the case when the mission is to have a given all-burnt velocity. Finally it is proved that in order to achieve a given all-burnt velocity by arranging the stages in parallel results in an increase in the total initial mass compared to the case when they are arranged in series and the magnitude of this increase depends upon the number of stages.

In a multi-stage rocket, for a given ratio of the vehicle gross mass to payload mass (defined as payload ratio), an infinite number of vehicle sizes and corresponding payloads that will perform a particular mission can be had. This ratio is, therefore, an important tool in the preliminary design for comparison purposes since it essentially represents the efficiency of the vehicle configuration for the design mission. The main problem for the designer of a multi-stage vehicle is, therefore, the minimisation of the payload ratio for a specified mission.

Interest in the study of the staging optimization started with the analysis of the relative size of multi-stage vehicle presented by Summerfield & Malina[1]. They obtained the solution of the problem under the restricted conditions that the specific impulses and structure factors (defined as the ratio of the structure mass to the mass of the stage under consideration) are constants and equal for each stage. Vertregt[2] extended the solution to the case when all the stages have unequal specific impulses. Goldsmith[3] further extended the results to the case of a two-stage rocket having different specific impulses and structure factors. Weisberd[4], Hall & Zambelli[5] presented another technique of solution when both specific impulses and structure factors may be different for each stage but do not vary with step size. Colemann[6] went further and showed that a better optimization analysis would be to include a scaling law for structure factor which accounts for its variation with step size. Chase[7] presented an analysis where the structure mass of a stage is related linearly to propellant mass. This relation was determined by actual preliminary design studies considering a given engine combination and a series of propellant leadings.

All these authors have either completely ignored the gravity or they have indicated to make allowances for it by suitably modifying the specific mission velocity. But it is always desirable to include this loss directly in the analysis than ignoring it all together or making some approximate allowance for it. The effect of gravity losses in the optimum distribution of total mass among the various stages by making the payload ratio minimum so that a specified velocity is attained at the end of powered phase has been studied in this paper. The main assumptions of the analysis are:

(*i*) the flight path is verticle,

(*ii*) the rocket vehicle is so large that the retarding effect of aerodynamic resistance may be neglected, (*iii*) there is no coasting period between the discarding of one stage and the firing of the next stage, and

(*iv*) thrusts, specific impulses, structural factors and accelerations due to gravity are constants but different for each stage.

It has been shown that in case when all the steps have constant physical parameters, i.e. have the same specific impulse, structure factors and weight to thrust ratio, the criterion for optimum staging is the same as given by Summerfield when gravity is neglected (the mass ratios must be the same for each step). The analysis has been carried out to the case when the mission requirement is to achieve a given all-burnt height. The analysis shows that even at the expense of more total weight, total height is not more than in the case when the mission is to have a given all-burnt velocity. Further it has been shown that to achieve a given all-burnt velocity by arranging the rockets in parallel always means an increase in the total weight than when they are arranged in series. The magnitude of this increase depends upon the number of stages.

BASIC EQUATIONS

The primary factors contributing to the velocity at the end of powered phase are: payload mass, propellant mass, structure mass, specific impulse, thrust and duration of burning period. Under the assumed conditions, the velocity attained at all-burnt is given by

$$V_b = \sum_{i=1}^{N} c_i \log r_i - \sum_{i=1}^{N} \frac{c_i \; g_i}{f_i} \; \frac{r_i - 1}{r_i} \tag{1}$$

where

$$r_i = \frac{M_{oi}}{M_{oi} - M_{pi}}, \; c_i = g_i \; I_{spi} \text{ and } f_i = F_i \; / M_{oi} \tag{2}$$

Now the payload ratio is given by

$$X = \frac{M_{oi}}{M_L} = \prod_{i=1}^{N} x_i \tag{3}$$

where x_i is the stage payload ratio defined as

$$x_i = \frac{M_{oi}}{M_{oi+1}}$$

and M_L is the payload mass.

In the absence of actual data it is quite reasonable to assume that the structure mass of a stage is proportional to take-off mass of the stage i.e.

$$\epsilon_i = \frac{M_{si}}{M_{si} + M_{pi}} = \frac{M_{si}}{M_{oi} - M_{oi+1}} \tag{4}$$

where ϵ_i is called the structure factor. It is not essential that the structure factors for the individual stages have identical values. The attainable values for ϵ_i are governed by the available constructional material, the knowledge of structural design and the ingenuity of the designer of the vehicle.

From (2), (3) and (4), we have

$$X = \prod_{i=1}^{N} \frac{r_i (1 - \epsilon_i)}{(1 - r_i \epsilon_i)} \tag{5}$$

CASE WHEN STRUCTURE FACTORS ARE CONSTANT

The problem consists of finding the minimum value of X so as to attain a specified mission, in this case, a specified all-burnt velocity and then for a given payload mass to find the distribution of total mass among the various stages. Thus (5) is to be minimised subject to condition that the vehicle attains a prescribed velocity at the end of the last stage of power flight. Instead of minimising X the analysis can be simplified by minimising log X, subject to constraint (1). Following the method of Lagrange's multiplier, the augmented function to be minimised is

$$G \equiv V_b - \sum_{i=1}^{N} c_i \log r_i + \sum_{i=1}^{N} \frac{c_i\, g_i}{f_i} \frac{r_i - 1}{r_i} + \gamma \sum_{i=1}^{N} \log \frac{r_i (1 - \epsilon_i)}{1 - r_i\, \epsilon_i} \tag{6}$$

where γ is the Lagrange multiplier to be determined.

The expressions giving the required values of r_i for minimum X are given by

$$\frac{g_i\, c_i}{f_i} \frac{1}{r_i} - c_i + \frac{\gamma}{1 - r_i\, \epsilon_i} = 0 \qquad i = 1, 2, \ldots, N \tag{7}$$

(7) and the condition (1) determine the (N+1) unknowns i.e. $N_{r_i's}$ and the multiplier γ. (7) gives the optimum sizing relationships in terms of mass ratios of each step. This can be rewritten as:

$$c_1 (1 - r_1 \epsilon_1) \left(\frac{g_1}{f_1 r_1} - 1 \right) = c_2 (1 - r_2\, \epsilon_2) (g_2/f_2 r_2 - 1) = \ldots\ldots$$

$$= c_N (1 - r_N\, \epsilon_N) \left(\frac{g_N}{f_N\, r_N} - 1 \right) \tag{8}$$

which reduces to the well-known relation

$$c_1 (1 - r_1 \epsilon_1) = c_2 (1 - r_2 \epsilon_2) = \ldots. = c_N (1 - r_N\, \epsilon_N)$$

when the gravity terms are dropped out.

In the special case when all the parameters of the step rockets are independent of i (i.e. $c_i = c_{i+1}$, $\epsilon_i = \epsilon_{i+1}$, $f_i = f_{i+1}$, $g_i = g_{i+1}$; $1 \leqslant i \leqslant (N-1)$)

we have

$$(1 - r_1 \epsilon_1) \left(\frac{g}{f r_1} - 1 \right) = (1 - r_2\, \epsilon_2) \left(\frac{g}{f r_2} - 1 \right) = \ldots = (1 - r_N\, \epsilon_N) \left(\frac{g}{f r_N} - 1 \right) \tag{9}$$

showing that the following can be had:

Case (i) All the mass ratios are equal to one another i.e.

$$r_1 = r_2 = \ldots\ldots r_N \tag{10}$$

Case (*ii*) m of the r_i 's are equal to one another and the rest $(N-m)$ r_i 's are equal to one another, where m can have any value from 1 to $(N-1)$ such that if r_1 and r_1^* represents the two sets then

$$r_1 \, r_1^* = \frac{g}{\epsilon f} \tag{10a}$$

In the case when gravity is neglected the condition as deduced by Summerfield is

$$r_1 = r_2 = \ldots\ldots = r_N$$

which is the same as (10) above.

In case (*i*) condition (10) is used and from (1) we get

$$V_b = NC \, \log\left(r_1 e^{-\frac{g}{f} \cdot \frac{r_1 - 1}{r_1}} \right) \tag{11}$$

which determines the mass ratio required in each step for the specified all-burnt velocity. Table 1 gives the values of r_1 for different values of V_b/NC and g/f

TABLE 1

VALUES OF r_1 FOR DIFFERENT VALUES OF V_b/NC AND g/f

V_b/NC	g/f					
	0	0·1	0·2	0·3	0·4	0·5
0·4	1·492	1·546	1·609	1·685	1·777	1·887
0·8	2·223	2·357	2·510	2·686	2·891	3·124
1·2	3·320	3·568	3·850	4·170	4·535	4·948
1·6	4·953	5·373	5·846	6·378	6·977	7·648

TABLE 2

VALUES OF $(X/X_o)^{\frac{1}{N}}$ FOR GIVEN VALUES OF V_b/NC AND g/f FOR THREE VALUES OF ϵ

$\frac{g}{f}$	ϵ	V_b/NC			
		0·4	0·8	1·2	1·6
·1	·05	1·039	1·067	1·091	1·116
	·10	1·043	1·077	1·116	1·183
	·15	1·047	1·091	1·161	1·437
·2	·05	1·085	1·146	1·198	1·265
	·10	1·093	1·170	1·260	1·445
	·15	1·103	1·250	1·378	2·465
·3	·05	1·141	1·239	1·323	1·422
	·10	1·156	1·283	1·440	1·794
	·15	1·173	1·346	1·684	7·644
·4	·05	1·210	1·349	1·473	1·628
	·10	1·223	1·420	1·670	2·352
	·15	1·260	1·527	2·145	—
·5	·05	1·292	1·478	1·651	1·881
	·10	1·326	1·587	1·971	3·313
	·15	1·369	1·759	2·902	—

The relation between mass-ratio and payload ratio in this case can be obtained from (5) and (10) as

$$X = \left[\frac{r_1 (1-\epsilon)}{1 - r_1 \epsilon}\right]^N \tag{12}$$

Here it is observed that the values of r_1 cannot be greater than $1/\epsilon$. Also the relation between all-burnt velocity and payload ratio is

$$\frac{X^{\frac{1}{N}}}{1-\epsilon\,(1-X^{\frac{1}{N}})} = e^{\frac{V_b}{NC} - \frac{g}{f}\,\frac{(1-\epsilon)(1-X^{\frac{1}{N}})}{X^{\frac{1}{N}}}} \tag{13}$$

If X_o represents the payload ratio when gravity is neglected then the values of $(X/X_o)^{1/N}$ are obtained from Table 2 for given values of $V_b\,/NC$ and g/f and three representative values of ϵ.

In case when the number of stages is very large, then the value of the ratio of gross-mass to payload mass in terms of specified all-burnt velocity is given by

$$X = e^{\frac{V_b}{C(1-\epsilon)\,(1-g/f)}} \tag{14}$$

Table 3 gives the values of X in this case for given values of V_b/C and g/f for three representative values of ϵ.

In case (*ii*) condition (10*a*) is to give the optimum staging and the value of r_1 is given by the relation

$$V_b = C \log \left[\left(\frac{g}{\epsilon f}\right)^m r_1^{(N-2m)}\; e^{-g/f\left\{N - r_1\left(\frac{m\epsilon f}{g} + \frac{N-m}{r_1^2}\right)\right\}} \right] \tag{15}$$

Also the payload ratio in terms of mass ratio is given by

$$X = \frac{(g/\epsilon f)^m\, r_1^{N-2m}(1-\epsilon)^N}{(1-r_1\epsilon)^{N-m}\,(1-g/f.\,1/r_1)^m} \tag{16}$$

TABLE 3

PAYLOAD RATIO IN CASE OF INFINITE STAGING FOR GIVEN VALUES OF Vb/C AND g/f FOR THREE VALUES OF ϵ

V_b/C	g/f ·1			·2			·3			·4		
	ϵ=0·5	·10	·15	ϵ=·05	·10	·15	ϵ=·05	·10	·15	ϵ=·05	·10	·15
0·5	1·794	1·854	1·922	1·931	2·003	2·086	2·121	2·211	2·317	2·404	2·524	2·666
1·0	3·218	3·437	3·694	3·729	4·012	4·351	4·498	4·889	5·368	5·779	6·371	7·102
1·5	5·774	6·373	7·100	7·200	8·036	9·077	9·542	10·809	12·439	13·893	16·079	18·927
2·0	10·358	11·815	13·646	13·904	16·906	18·935	20·238	23·898	28·821	33·399	40·584	50·442
2·5	18·583	21·905	26·228	26·848	32·241	39·498	42·924	52·837	66·778	80·292	102·435	134·427

Now in what follows, if the symbol (′) denotes the values corresponding to condition (10), then for the same value of r_1 in both the cases (10) and (10a), we have from (12) and (16)

$$\frac{X'}{X} = \left[\frac{\epsilon f}{g}\;\frac{r_1(r_1 - g/f)}{1 - r_1\epsilon}\right]^m \tag{17}$$

From (17) it is clear that we require more initial gross mass for the same payload by using condition (10) than by using condition (10a) provided that

$$r_1 > \sqrt{\frac{g}{\epsilon f}} \tag{18}$$

Further we see that

$$V_b' - V_b = Cm\left[\log\frac{r_1}{r_1^*} - m\epsilon r_1\left(1 - \frac{g}{\epsilon f}\;\frac{1}{r_1^2}\right)\right]$$

and $V_b' > V_b$ only if

$$\log\frac{r_1}{r_1^*} > \epsilon r_1\left(1 - \frac{g}{\epsilon f}\;\frac{1}{r_1^2}\right)$$

and this will be so only when

$$r_1 > \sqrt{\frac{g}{\epsilon f}}$$

Therefore for a given payload mass, more initial gross mass is required and at the same time get more velocity at the end of powered flight by using condition (10) provided (18) is satisfied, otherwise the case is opposite. Obviously the two conditions (10) and (10a) are identical if $r_1 = \sqrt{g/f\epsilon}$ At this point the common value of V_b and X are given by

$$\frac{V_b}{NC} = \log\sqrt{\frac{g}{\epsilon f}} - \frac{g}{f}\left(1 - \sqrt{\frac{\epsilon f}{g}}\right) \tag{19}$$

and

$$X^{\frac{1}{N}} = \frac{1 - \epsilon}{\sqrt{\frac{\epsilon f}{g}} - \epsilon} \tag{20}$$

Here it is seen that for given ϵ, V_b/CN is maximum when

$$\frac{f}{g} = \left(\frac{\epsilon}{2} + 2\right) \pm \sqrt{\left(\frac{\epsilon}{2}\right)^2 + 2\epsilon}$$

Now in what follows it will be shown that the condition (10a) is true in a limited sense for optimum staging but has got the advantage that even for a nominal increase in total mass all-burnt velocity is obtained at a considerably larger all-burnt height than is given by condition (10). Since both the conditions are to give the specified all-burnt velocity, from (11) and (15) it follows that

$$\frac{g}{f} = \frac{N\log\frac{r_1'}{r_1} + m\log\frac{r_1}{r_1^*}}{\frac{m}{r_1^*} - \frac{N}{r_1'} + \frac{N-m}{r_1}} \tag{21*}$$

If condition (10a) is to be a better optimum staging criterion than (10), (12) and (16) give that

$$\frac{r_1'^N}{r_1^{N-m}\, r_1^{*m}} > \frac{(1-r_1'\epsilon)^N}{(1-r_1\epsilon)^{N-m}\,(1-r_1^*\epsilon)^m} \tag{22}$$

Therefore in order that condition (10a) is to have an edge over condition (10), then (21), (22) and (10a) must be simultaneously satisfied. For the case $m=\frac{N}{2}$, it is easy to show from (22) that

$$\epsilon < \frac{r_1'^2 - r_1 r_2}{r_1'^2\,(r_1+r_1^*) - 2r_1\, r_1^* r_1'} \tag{23}$$

But from (10a) and (21)

$$\epsilon = \frac{1}{r_1 r_1^*}\ \frac{\log \dfrac{r_1'^2}{r_1 r_1^*}}{1/r_1 + 1/r_1^* - \dfrac{2}{r_1'}} \tag{24}$$

and therefore from (23) we should have

$$\log \frac{r_1'^2}{r_1\, r_1^*} < \left(1 - \frac{r_1\, r_1^*}{r_1'^2}\right)$$

and this cannot be possible. This shows that condition (10) which is the Summerfield criterion is only true for optimum staging. But if we consider the example where

$$\epsilon = \cdot 041, \qquad g/f = \cdot 655, \qquad V_b = 20000 \text{ ft/sec.},$$
$$M_L = 1000 \text{ lbs.}, \qquad C = 8000 \text{ ft/sec.} \quad \text{and} \quad N = 2,$$

we see that we require about 2640 lbs more by using condition (10a) but get the all-burnt velocity at a height greater by 113325 ft.

It is seen that in general equation (7) relates the parameter of each stage of a vehicle for calculating the optimum sizing. But those equations in themselves are not sufficient since we have to attain a given mission, i.e. a specified velocity at the end of powered flight. Therefore we must solve (1) and (7) simultaneously for r_i 's and γ. This can be done as follows :

(7) can be re-written as

$$c_i\ \epsilon_i\ r^2_i - \left[c_i\ \left(1+\frac{g_i\ \epsilon_i}{f_i}\right) - \gamma\right] r_i + \frac{c_i\ g_i}{f_i} = 0$$

therefore

$$r_i = \frac{\left\{c_i\ \left(1+\frac{g_i\ \epsilon_i}{f_i}\right) - \gamma\right\} \pm \sqrt{\left\{c_i\ \left(1+\frac{g_i\ \epsilon_i}{f_i}\right) - \gamma\right\}^2 - 4c_i\ \epsilon_i\ \frac{g_i\ c_i}{f_i}}}{2c_i\ \epsilon_i}$$

*Since g/f is to be less than unity it can be shown that for $m=N/2$ and for known values of r_1 and r_1^* the possible value of r'_1 must be so as to satisfy $G < r'_1 < \frac{A\,(H-1)}{G-1}$ where A, G and H are the arithmatic, geometric and harmonic means respectively of r_1 and r^*_1.

The question of sign is decided by the fact that when gravity terms are ignored the values of r_i should be positive and so we have to choose the positive sign. Thus

$$r_i = \left(a_i - \frac{\gamma}{2c_i \ \epsilon_i}\right) + \sqrt{\left(a_i - \frac{\gamma}{2c_i \ \epsilon_i}\right)^2 - b_i} \tag{25}$$

where

$$a_i = \frac{f_i + g_i \ \epsilon_i}{2f_i \ \epsilon_i}, \ b_i = \frac{g_i}{f_i \ \epsilon_i}$$

Therefore

$$V_b = \sum_{i=1}^{N} c_i \log \left\{ \left(a_i - \frac{\gamma}{2 c_i \ \epsilon_i}\right) + \sqrt{\left(a_i - \frac{\gamma}{2c_i \ \epsilon_i}\right)^2 - b_i} \right\}$$
$$- \sum_{i=1}^{N} \left\{ 1 - \frac{1}{\left(a_i - \frac{\gamma}{2c_i \ \epsilon_i}\right) + \sqrt{\left(a_i - \frac{\gamma}{2c_i \ \epsilon_i}\right)^2 - b_i}} \right\} \tag{26}$$

This is a transcendental equation in one unknown γ and can be solved easily by any of the well-known methods such as Newton-Rapheon's method or the iterative method. Having evaluated γ and knowing the values of r_i 's from (7) we can calculate the masses of the different stages for a given mass of the payload. As a sample example and for the ease in calculation let us consider the following case of a two-stage rocket.

$c_1 = 8000$ ft/sec.	$g_1/f_1 = 0 \cdot 5$	$\epsilon_1 = 0 \cdot 15$	$V_b = 17300$ ft/sec.
$c_2 = 10000$ ft/sec.	$g_2/f_2 = 0 \cdot 4$	$\epsilon_2 = 0 \cdot 20$	$M_L = 2000$ lbs.

Tuble 4 gives the mass distribution in the two stages and gives the figures both when gravity terms are neglected as well as when it is accounted for.

Table 4 shows the sensitivity of the mass distribution in the two stages to the gravity terms. We require an initial gross mass of about five times when gravity is accounted for directly than when it is neglected all together. This shows that a more realistic picture of the optimum staging can be had only when we consider the gravity losses also.

TABLE 4

MASS DISTRIBUTION IN VARIOUS STAGES FOR ZERO AND NON-ZERO GRAVITY

	Mass distribution (in lbs).	
	Gravity included	Gravity neglected
Mass of first step Mo_1	134535	27152
Mass of propellant in first stage Mp_1	100283	16898
Structure mass of first stage Ms_1	17698	2982
Mass of second step Mo_2	16554	7272
Mass of propellant in second stage Mp_2	11643	5272
Structure mass of second stage Ms_2	2911	1318

Now we discuss the mass distribution in various stages for minimum payload ratio when the constraint to be used is to obtain a specified height at all-burnt. This height is given by the expression

$$h_b = \sum_{i=1}^{N} \frac{c^2_i}{f_i} \frac{r_i - 1}{r_i} \left\{1 + \frac{\log r_i}{1 - r_i}\right\} - \tfrac{1}{2} \sum_{i=1}^{N} \frac{g_i\, c^2_i}{f^2_i} \left(\frac{r_i - 1}{r_i}\right)^2 \tag{27}$$

Therefore from (5) and (27), we have to minimise the augmented function

$$G \equiv h_b - \sum_{i=1}^{N} \frac{c^2_i}{f_i} \frac{r_i - 1}{r_i} \left\{1 + \frac{\log r_i}{1 - r_i}\right\} + \tfrac{1}{2} \sum_{i=1}^{N} \frac{g_i\, c^2_i}{f^2_i} \left(\frac{r_i - 1}{r_i}\right)^2 + \phi \sum_{i=1}^{N} \log \frac{r_i\,(1 - \epsilon_i)}{1 - r_i\, \epsilon_i}$$

The values of r_i 's giving the minimum value of X are given by

$$g_i \frac{c^2_i}{f^2_i} \frac{r_i - 1}{r_i} - \frac{c^2_i}{f_i} \frac{\log r_i}{r_i} + \frac{\varphi}{1 - r_i\, \epsilon_i} = 0 \qquad i = 1, 2, .., N \tag{28}$$

These N equations together with (27) determine the values of r_i's and φ. Knowing the values of r_i's we can calculate the masses of the various stages for a given payload mass. The procedure to be followed for the solution can be as follows:

From (28) we have

$$\frac{c^2_i}{f_i} \left(1 - r_i\, \epsilon_i\right) \left(\frac{g_i}{f_i} \frac{r_i - 1}{r_i} - \frac{\log r_i}{1 - r_i} \right) = \frac{c^2_{i+1}}{f_{i+1}} \left(1 - r_{i+1}\epsilon_{i+1}\right) \times$$

$$\left(\frac{g_{i+1}}{f_{i+1}} \frac{r_{i+1} - 1}{r_{i+1}} - \frac{\log r_{i+1}}{1 + r_{i+1}} \right) \qquad 1 \leqslant i \leqslant (N - 1) \tag{29}$$

Assume some value of r_i and iterate on r_{i+1} till the above equation is satisfied. After finding r_{i+1} iterate on r_{i+2} and so on. After having found all the values of r_i's put those values in (27) to test the fulfilment of the constraint. Iterate on r_i and repeat the process till the constraint is satisfied.

This method is adopted for the two-stage missile whose data is given earlier. The minimum initial gross mass and its distribution for a given all-burnt velocity have been already found. With that distribution the all-burnt height is 563130 ft. In order to compare the two cases take this all-burnt height as our mission requirement and find out the total minimum mass required and also its mass break up in the two stages. The mass distribution in this case comes out to be as follows:

	(lbs.)
Mass of first step	157215
Propellant mass in first stage	107100
Structure mass in first stage	18900
Mass of second step	31215
Propellant mass of second stage	23372
Structure mass of second stage	5843

For this distribution the all-burnt velocity is 17230 ft/sec. which is less than 17300 ft/sec as the mission velocity earlier. If it is compared with Table 4, it is noticed that more initial gross mass is required in this case and still get less all-burnt velocity and so also less total height. This shows that even at the expanse of having more weight we are not getting more height than in the earlier case. Hence it is always advisable to optimise the all-burnt velocity for optimum staging.

CASE WHEN STRUCTURE FACTOR IS VARIABLE

As stated earlier Coleman pointed out the utility of including a scaling law for structure factor which accounts for its variation with step size. Chase took one such relation as

$$M_{si} = A_i \; M_{pi} + B_i \tag{30}$$

The coefficients A_i 's and B_i 's are determined by preliminary design studies for each step corresponding to the propellants and thrust levels under consideration. Chase's analysis is very cumbersome and does not give any explicit relationships giving the optimum conditions. Following the above relation (30) the analysis is carried out in details and those conditions are obtained explicitly. Since

$$M_{pi} = (M_{oi} - M_{oi+1}) - M_{si} \tag{31}$$

From (30) and (31), we have

$$M_{pi} = \frac{(M_{oi} - M_{oi+1}) - B_i}{(1 + A_i)} \tag{32}$$

Therefore

$$r_i = \frac{M_{oi}(1 + A_i)}{M_{oi} A_i + M_{oi+1} + B_i} = \frac{x_i \; (1 + A_i)}{x_i \; A_i + 1 + \dfrac{B_i}{M_{oi+1}}} \tag{33}$$

$$t_{bi} = \frac{c_i}{\alpha_i} \; \frac{M_{pi}}{M_{oi}} = \frac{c_i}{\alpha_i \; (1 + A_i)} \; \frac{x_i - 1 - \dfrac{B_i}{M_{oi+1}}}{x_i} \tag{34}$$

Now the following two cases are discussed :

(*i*) when the total initial mass is given

(*ii*) when the payload mass is given.

Case (*i*) In this case, since

$$M_{oi+1} = \frac{M_{oi}}{\prod_{k=1}^{i} x_k}$$

therefore all-burnt velocity is given by

$$V_b = \sum_{i=1}^{N} c_i \log \frac{x_i \; (1 + A_i)}{x_i \; A_i + 1 + \dfrac{B_i}{M_{oi}} \prod_{k=1}^{i} x_k} - \sum_{i=1}^{N} \frac{c_i \; g_i}{\alpha_i \; (1 + A_i)} \times \frac{x_i - 1 - \dfrac{B_i}{M_{oi}} \prod_{k=1}^{i} x_k}{x_i} \tag{35}$$

We have to minimising $\log X$ i.e., $\sum_{i=1}^{N} \log x_i$ subject to condition (35). Following the usual principle of Lagrenge's multiplier, we have the required conditions as

$$\frac{g_i\ C_i}{\alpha_i\ (1+A_i)}\ \frac{1}{x_i} - \frac{C_i}{x_i\ A_i + \frac{B_i}{M_{o1}} \prod_{k=1}^{i} x_k} + \varphi = 0 \qquad i = 1, 2, \ldots\ldots, N$$

or

$$C_i \left[\frac{g_i}{\alpha_i\ (1+A_i)}\ \frac{1}{x_i} - \frac{1}{x_i\ A_i + 1 + \frac{B_i}{M_{o1}} \prod_{k=1}^{i} x^k} \right] = C_{i-1} \left[\frac{g_{i-1}}{\alpha_{i-1}\ (1 + A_{i-1})\ x_{i-1}} \right.$$

$$\left. - \frac{1}{x_{i-1}\ A_{i-1} + 1 + \frac{B_{i-1}}{M_{o1}} \prod_{k=1}^{i-1} x_k} \right] \qquad 2 \leqslant i \leqslant N \qquad (36)$$

In order to solve (36) first choose some appropriate value of x_{i-1} and iterate on x_i. After having found x_{i-1} and x_i, iterate on x_{i-2} and so on. Having found all the values of $x_{i's}$, put in (35) to satisfy the constraint. After knowing all the values of $x_{i's}$, the individual masses of the different stages are found out.

If the conditions (36) are expressed in terms of r_i and ϵ_i it is observed that the modified conditions are

$$\frac{C_1}{(1\epsilon_1) - (1+A_1)} (1-r_1\ \epsilon_1)\left(\frac{g_1}{\alpha_1}\ \frac{1}{r_1} - 1\right) = \frac{C_2}{(1-\epsilon_2)\ (1+A_2)} (1-r_2\ \epsilon_2)\left(\frac{g_2}{\alpha_2}\ \frac{1}{r_2} - 1\right) =$$

$$\ldots = \left(\frac{C_N}{(1-\epsilon_N)\ (1+A_N)}\ (1-r_N\ \epsilon_N)\right)\left(\frac{g_N}{\alpha_N}\ \frac{1}{r_N} - 1\right) \qquad (37)$$

In the special case when the different physical parameters (*i.e.* $A_i, B_i, C_i, \epsilon_i, g_i/\alpha_i$) are independent of i, the same conditions as (10) and (10a) are obtained.

Case (ii) — In this case

$$M_{o\,i+1} = M_L \prod_{k=i+1}^{N} x_k \qquad (38)$$

therefore all-burnt velocity is expressed as

$$V_b = \sum_{i=1}^{N} C_i \log \frac{x_i\ (1+A_i)}{x_i\ A_i + 1 + \frac{B_i}{M}\left(\prod_{k=i+1}^{N} x_k\right)^{-1}} - \sum_{i=1}^{N} \frac{g_i\ C_i}{\alpha_i\ (1+A_i)}\ \frac{x_i - 1 - \frac{B_i}{M_L}\ \frac{1}{\prod_{k=i+1}^{N} x_k}}{x_i} \qquad (39)$$

and the conditions corresponding to (36) are

$$C_i \left(1 + \frac{B_i}{M_L}\ \frac{1}{\prod_{k=i+1}^{N} x_k}\right) \left[\frac{g_i}{\alpha_i (1+A_i)}\ \frac{1}{x_i} - \frac{1}{x_i\ A_i + 1 + \frac{B_i}{M_L}\ \frac{1}{\prod_{k=i+1}^{N} x_k}} \right]$$

$$= C_{i-1}\left(1+\frac{B_{i-1}}{M_L}\,\frac{1}{\prod_{k=i}^{N} x_k}\right)\left[\frac{g_{i-1}}{\alpha_{i-1}\,(1+A_{i-1})\,x_{i-1}}\,\frac{1}{x_{i-1}}-\frac{1}{x_{i-1}\,A_{i-1}+1+\frac{B_{i-1}}{M_L}\cdot\frac{1}{\prod_{k=i}^{N} x_k}}\right]$$

$$2 \leqslant i \leqslant N \qquad (40)$$

Applying the usual iteration technique to solve these equations to get the values of $x_{i's}$ which also satisfy the given constraint (39). Here again if the conditions (40) are expressed in terms of $r_{i's}$ we have

$$\frac{C_1}{1+A_1}\{1+A_1(1-r_1)\}\left\{\frac{g_1}{\alpha_1}\,\frac{1}{r_1}-1\right\}=\frac{C_2}{1+A_2}\{1+A_2(1-r_2)\}\left\{\frac{g_2}{\alpha_2}\,\frac{1}{r_2}-1\right\}$$

$$=\ldots\ldots\ldots=\frac{C_N}{1+A_N}\{1+A_N(1-r_N)\}\left\{\frac{g_N}{\alpha_N}\,\frac{1}{r_N}-1\right\} \qquad (41)$$

Also in the special case when $C_i = C_{i+1}$, $A_i = A_{i+1}$, $\frac{g_i}{\alpha_i}=\frac{g_{i+1}}{\alpha_{i+1}}$ $\{1 \leqslant i \leqslant (N-1)\}$

the following two conditions are obtained : (42)

Condition (*i*) $r_1 = r_2 = \ldots\ldots\ldots\ldots = r_N$

Condition (*ii*) $m r_{i's}$ are equal and $(N-m)\; r^*_{i\,s}$ are equal such that

$$r_i\; r^*_i = \left(1+\frac{1}{A_1}\right)\frac{g}{\alpha} \qquad (42a)$$

CASE OF PARALLEL STAGING

Another device of staging rockets consists in arranging them in parallel instead of in series as done above. In this case all the engines are used simultaneously to give their full thrusts from the very beginning, instead of being fired in succession. Quite often parallel staging is proposed in order to achieve better performance than is obtained with series staging. Here again the total payload ratio is the product of the payload ratios of the individual stages, i.e.

$$X = \prod_{i=1}^{N} x_i \qquad (43)$$

It is assumed here that the engine weight of any stage is proportional to the maximum thrust of the engine. Thus if F_i is the total thrust in any stage then the engine weight is given by

$$M_{ei} = \lambda_i\left(\alpha_i - \frac{\alpha_{i+1}}{x_i}\right)M_{oi} \qquad (44)$$

We define another ratio as

$$\mu_i = \frac{M_{si}}{M_{pi}} \qquad (45)$$

since

$$M_{oi} = M_{pi} + M_{si} + M_{ei} + M_{oi+1} \qquad (46)$$

Therefore from (44), (45), and (46), we have

$$M_{pi} = \frac{(M_{oi} - M_{oi+1}) - \left(\alpha_i - \frac{\alpha_{i+1}}{x_i}\right)\lambda_i\, M_{oi}}{1+\mu_i} \qquad (47)$$

Now in this case also the all-burnt velocity is

$$V_b = \sum_{i=1}^{N} C_i \log r_i - \sum_{i=1}^{N} \frac{C_i g_i}{\alpha_i} \frac{r_i - 1}{r_i} \tag{48}$$

where

$$r_i = \frac{x_i (1 + \mu_i)}{1+\mu_i x_i + \left(\alpha_i - \frac{\alpha_{i+1}}{x_i}\right)\lambda_i x_i} = \frac{x_i (1+\mu_i)}{(1- \lambda_i \alpha_{i+1}) + x_i (\mu_i + \lambda_i \alpha_i)}$$

Here again (43) is to be minimised in order to achieve a given all-burnt velocity given by (48). Following the usual Lagrangian method, we have

$$\frac{C_i g_i}{\alpha_i} \frac{1-\lambda_i \alpha_{i+1}}{(1+\mu_i)x_i} - \frac{C_i (1-\lambda_i \alpha_{i+1})}{(1-\lambda_i \alpha_{i+1}) + x_i (\mu_i + \lambda_i \alpha_i)} + \phi = O \tag{49}$$

where ϕ is the Lagrangian multiplier.

Since for the Nth stage $\alpha_{N+1}= O$, we have

$$\frac{C_N g_N}{\alpha_N} \frac{1}{(1+\mu_N)x_N} - \frac{C_N}{1 + x_N (\mu_N + \lambda_N \alpha_N)} + \phi = O \tag{50}$$

Therefore from (49) and (50), we obtain

$$\frac{C_i (1- \lambda_i \alpha_i)}{(1- \lambda_i \alpha_{i+1}) + x_i (\mu_i + \lambda_i \alpha_i)} - \frac{C_i g_i}{\alpha_i} \frac{1-\lambda_i \alpha_{i+1}}{(1+\mu_i)x_i}$$

$$= \frac{C_N}{1 + x_N (\mu_N + \lambda_N \alpha_N)} - \frac{C_N g_N}{\alpha_N} \frac{1}{(1+\mu_N) x_N} \tag{51}$$

$$i=1,2,\ldots\ldots\ldots\ldots, (N-1)$$

If we express this condition in terms of mass ratio r_i we can simplify it as

$$C_i \left(\frac{g_i}{\alpha_i} - r_i\right) \left(\frac{1}{r_i} - \frac{\mu_i + \lambda_i \alpha_i}{1+\mu_i}\right) = C_N \left(\frac{g_N}{\alpha_N} - r_N\right) \left(\frac{1}{r_N} - \frac{\mu_N + \lambda_N \alpha_N}{1+\mu_N}\right)$$

$$i = 1, 2, \ldots\ldots, (N - 1) \tag{52}$$

These equations can be solved for different r_i's by the well-known iteration technique and knowing all the values of r_i's the minimum initial gross mass as well as its distribution in the various stages are known.

Here again, as earlier, we observe that in the special case when all the parameters $\lambda_i, \mu_i, \frac{g_i}{\alpha_i}$ and C_i are the same for each stage we have

Condition (*i*) $r_1 = r_2 = \ldots\ldots\ldots = r_N$ (53)

Condition (*ii*) mr_i's are equal and $(N-m)$ r^*_i's are equal such that

$$r_i r_i^* = \frac{g}{\alpha} \frac{1 + \mu}{\mu + \lambda \alpha} \tag{53a}$$

and as earlier for the case of series staging it can also be shown here that condition (53) is true for optimum staging. In this special case the total payload ratio is expressed by

$$X = \left[\frac{r_i(1-\lambda\alpha)}{(1+\mu)-r_i\ (\mu+\lambda\alpha)}\right]^N \frac{1}{1-\lambda\alpha} \tag{54}$$

Also (12) gives the total payload ratio required to achieve the same all-burnt velocity when the different stages are arranged in series. Therefore from (12) and (54), we have

$$\left[\frac{(1-\lambda\alpha)X}{X'}\right]^{1/N} = \frac{(1-\lambda\alpha)\ (1-r_1\,\epsilon)}{\{(1+\mu)-r\,(\mu+\lambda\alpha)\}\ (1-\epsilon)} \tag{55}$$

Using (4) and (45) this can be re-written as

$$\left[\frac{(1-\lambda\alpha)\ X}{X'}\right]^{1/N} = \frac{(1-\lambda\alpha)\ (1-r\epsilon)}{(1-r\,\epsilon)\ -\ r\,\lambda\alpha\ \ (1-\epsilon)}$$

But from (44) $\lambda\alpha < 1$ and so for the same payload to achieve the same all-burnt velocity we require less mass in the case of parallel staging only if

$$1 < r < 1-(1-\lambda\alpha)^{1-1/N} / \epsilon + \lambda\alpha(1-\epsilon) - \epsilon(1-\lambda\alpha)^{1-1/N}$$

which is contradictory to the fact. Hence there is no distinct advantage in weight saving in the case when the different stages are arranged in parallel.

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