UNSTEADY HEAT TRANSFER FROM A NON-ISOTHERMAL ROTATING DISC

BAL KRISHAN

Defence Science Laboratory, Delhi

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An analytic solution of the energy equation is obtained for a non-isothermal disc rotating in an incompressible fluid at rest neglecting the viscous dissipation. Initially the disc and the fluid are at a common temperature. Without altering the velocity, the disc temperature is then changed and maintained at at tmperature varying according to the power law of radial distance. Expressions for temperature distribution in the fluid for large and small times have been evaluated.

The flow, induced by a disc rotating in an incompressible fluid at rest, was frostconsidered by Von Karman[1] and later by Cochran[2]. The associated heat transfer problem, with the disc maintained at a constant temperature, was studied by Millisaps & Pohlhausen[3]. Morgan & Warner[4] obtained an approximate solution both for constant and arbitrary radial plate temperatures, using Lighthill's[5] approximation in which the convection velocity profile was approximated by its linear form. Recently, the problem in its steady state has received considerable attention and the solutions have been obtained under less restrictive conditions. In a recent work on the thermal boundary layer on a flat plate. Riley[6] has suggested that for sufficiently small values of time the thickness of the thermal boundary layer is small irrespective of the values of Prandtl number and the convection is affected by the velocity components near the wall which should therefore be replaced by their values near the wall. For solutions valid for large times Riley has shown that the departures from steady state are maximum near the wall and therefore near the steady state. The velocity components are again replaced by their values near the wall. In this paper we follow Riley's approach to discuss the transient thermal boundary layer on a rotating disc. Initially the disc and the surrounding incompressible fluid are assumed to be at a common temperature such that there exists only the usual velocity boundary layer and no thermal boundary layer. The thermal boundary layer appears when the disc temperature is raised to a temperature varying with radius in a power law. It is assumed that the effect of increase in disc temperature on the density of the fluid is negligible and the consequent effect on already established boundary layer can also be neglected. An analytic solution of the energy equation for small values of time has been obtained and it has been shown that steady state is approached as an exponential decay.

PROBLEM

The physical model is that of a large disc in the plane $z = 0$ rotating, in an incompressible viscous fluid at rest, around the axis $r = 0$ with angular velocity ω. Initially the disc and the fluid are at common temperature T_∞. At time $t = 0$ the temperature of the disc is changed to $T_\omega = T_\infty + nr^m$ where n and m are constants. Taking into consideration the boundary layer approximations, axial symmetry and neglecting the viscous dissipation the energy equation for such a system may be written in cylindrical coordinates as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial z^2} \qquad (1)$$

Here u and w respectively are the radial and axial components of velocity and k is the thermal diffusivity assumed to be constant, and is of order $(\delta_T)^2$ where δ_T is the thickness of thermal boundary layer.

Boundary and initial conditions are

$$\left.\begin{array}{llll} u = 0 & w = 0 & & z = 0 \\ u = 0 & & & z = \infty \\ T = T_\infty & & t = 0 & z \geqslant 0 \\ T = T_w = T_\infty + nr^m & & t \geqslant 0 & z = \infty \\ T \rightarrow T_\infty & & & z \rightarrow \infty \end{array}\right\} \quad (2)$$

Introducing the transformations

$$\left.\begin{array}{l} z = \sqrt{\dfrac{\nu}{\omega}}\,\zeta\,, \quad u = r\,\omega\,F(\zeta) \quad w = \sqrt{\nu\omega}\,H(\zeta) \\ t = \tau/\omega \quad \text{and} \quad \dfrac{T - T_\infty}{T_w - T_\infty} = \theta(\zeta, \tau) \end{array}\right\} \quad (3)$$

where ζ, F, H, τ and θ are dimensionless variables, equation (1) becomes

$$\frac{\partial^2\theta}{\partial\zeta^2} - \sigma H \frac{\partial\theta}{\partial\zeta} - \sigma F m \theta - \sigma \frac{\partial\theta}{\partial\tau} = 0 \quad (4)$$

where $\sigma = \nu/k$ is the Prandtl number. In the initial stages of growth of thermal boundary layer when the effects of thermal diffusion are important we define a new independent variable

$$\xi = \tfrac{1}{2} \frac{z}{(kt)^{\frac{1}{2}}} = \frac{\zeta}{2}\left(\frac{\sigma}{\tau}\right)^{\frac{1}{2}}$$

The energy equation (4) now becomes

$$\frac{\partial^2\theta}{\partial\xi^2} - \left\{2H(\sigma\tau)^{\frac{1}{2}} - 2\xi\right\}\frac{\partial\theta}{\partial\xi} - 4\tau F m \theta - 4\tau\frac{\partial\theta}{\partial\tau} = 0 \quad (5)$$

The boundary and initial conditions are accordingly transformed to

$$\left.\begin{array}{lll} F = 0 & H = 0 & \xi = 0 \\ F = 0 & & \xi = \infty \\ \theta = 0 & \tau = 0 & \xi \geqslant 0 \\ \theta = 1 & \tau > 0 & \xi = 0 \\ \theta \rightarrow 0 & \tau > 0 & \xi \rightarrow \infty \end{array}\right\} \quad (6)$$

Solution for small τ

As already mentioned we assume that for sufficiently small values of τ the thermal boundary layer, growing within the already established steady velocity boundary layer is very much thinner than the latter. The velocity components F and H may therefore be replaced by their values near the wall. These values in their series expansions as given by Cochran around $\zeta = 0$ may be written in terms of ξ and τ as

$$F = 2a_o\,\xi\left(\frac{\tau}{\sigma}\right)^{\frac{1}{2}} - 2\xi^2\left(\frac{\tau}{\sigma}\right) - \frac{8}{3}b_o\,\xi^3\left(\frac{\tau}{\sigma}\right)^{\frac{3}{2}} - \frac{4}{3}b_o^2\,\xi^4\left(\frac{\tau}{\sigma}\right)^2 - \frac{8a_o}{15}\xi^5\left(\frac{\tau}{\sigma}\right)^5$$

and

$$H = -4a_o\,\xi^2\left(\frac{\tau}{\sigma}\right) + \frac{8}{3}\xi^3\left(\frac{\tau}{\sigma}\right)^{\frac{3}{2}} + \frac{8}{3}b_o\,\xi^4\left(\frac{\tau}{\sigma}\right)^2 + \frac{16}{15}\xi^5\left(\frac{\tau}{\sigma}\right)^{\frac{5}{2}} + \frac{16}{45}a_o\,\xi^6\left(\frac{\tau}{\sigma}\right)^3 \qquad (7)$$

Substituting these values in (5) we get after rearranging

$$\frac{\partial^2\theta}{\partial\xi^2} + 2\xi\frac{\partial\theta}{\partial\xi} - 4\tau\frac{\partial\theta}{\partial\tau} = \frac{a_o\,\tau^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}}\left\{-8\xi^2\frac{\partial\theta}{\partial\xi} + 8\xi m\,\theta\right\}$$

$$+\frac{\tau^2}{\sigma}\left\{\frac{16}{3}\xi^3\frac{\partial\theta}{\partial\xi} - 8\xi^2 m\theta\right\} + \frac{b_o\,\tau^{\frac{5}{2}}}{\sigma^{\frac{3}{2}}}\left\{\frac{16}{3}\xi^4\frac{\partial\theta}{\partial\xi} - \frac{32}{3}\xi^3 m\theta\right\}$$

$$+\frac{b_o^2\,\tau^3}{\sigma^2}\left\{\frac{32}{15}\xi^5\frac{\partial\theta}{\partial\xi} - \frac{16}{3}\xi^4 m\theta\right\} \qquad (8)$$

We consider the possibility of the solution of this equation in the form

$$\theta(\xi,\tau) = \sum_{r=0}^{\infty}\tau^{\frac{3}{2}r}\,\phi_r(\xi) \qquad (9)$$

where r is an integer and $\phi_r(\xi)$ satisfy ordinary differential equations

$$\phi_o'' + 2\xi\phi_o' = 0 \qquad (10)$$

$$\phi_1'' + 2\xi\phi_1' - 6\phi_1 = \frac{8\xi a_o}{\sigma^{\frac{1}{2}}}\left\{m\phi_o - \xi\phi_o'\right\} \qquad (11)$$

$$\phi_2'' + 2\xi\phi_2' - 12\phi_2 = \frac{8\xi a_o}{\sigma^{\frac{1}{2}}}\left\{m\phi_1 - \xi\phi_1'\right\} + \frac{b_o^2\,\xi^4}{\sigma^2}\left\{\frac{32}{15}\xi\phi_o' - \frac{16}{3}m\phi_o\right\} \qquad (12)$$

with boundary conditions

$$\phi_o(0) = 1 \qquad \phi_r(0) = 0 \qquad r>0$$
$$\phi_r(\infty) = 0 \qquad \text{for all } r \qquad (13)$$

Solution of (10) to (12) subject to (13) are

$$\phi_o = \operatorname{erf} c\,\xi \qquad (14)$$

$$\phi_1 = \frac{a_o}{3\sqrt{\pi\sigma}}\left[\frac{\pi}{2}\operatorname{erf} c\,\xi\left\{2(3m-1)\xi^3 - 3(m+1)\xi\right\} - 3(m+1)\xi^2 e^{-\xi^2}\right] \qquad (15)$$

$$\phi_2 = \frac{1}{c} \left| \frac{e^{-\xi^2}}{\pi} \left[\left\{ 9\sqrt{\pi} \left(\frac{5}{2} a_1 + \frac{3}{2} a_2 \right) + 6(3b_1 + 7b_2) \right\} \xi + \right. \right.$$
$$+ \left\{ 12\sqrt{\pi} (5a_1 + 6a_2) + 16 (3b_1 + 7b_2) \right\} \xi^3 +$$
$$\left. + \left\{ \sqrt{\pi} (60a_1) + 16 (3b_1 + b_2) \right\} \xi^5 \right] -$$
$$\left. - \frac{8 \text{ erf } c\xi}{\sqrt{\pi}} \left\{ 15b_2\xi^4 + 2(3b_1 + b_2)\xi^6 \right\} \right| \quad (16)$$

where
$$C = -\frac{30 \times 48}{\sqrt{\pi}}$$

$$a_1 = -\frac{64}{15\sigma\sqrt{\pi}} \left\{ 5a^2_o + \frac{b^2_o}{\sigma} \right\} ; \; a_2 = -\frac{8a^2_o}{\sigma\sqrt{\pi}} (m^2 - 1)$$

$$b_1 = \frac{8}{3\sigma} \left\{ (3m^2 - 10m + 3) a^2_o - \frac{2mb^2_o}{\sigma} \right\} ; \; b_2 = -\frac{4a^2_o}{\sigma} (m^2 - 1)$$

From the definition of heat transfer coefficient h_r ,

$$-K \left(\frac{\delta T}{\delta z} \right)_{z=0} = hr \, (T\omega - T_\infty)$$

$$hr = -K \left(\frac{\omega}{\nu} \right)^{1/2} \left(\frac{\delta \theta}{\delta \zeta} \right)_{\zeta = 0}$$

where K is the thermal conductivity of the fluid. Then a local Nusselt number may be defined as

$$Nu_r = \frac{hr}{K} \left(\frac{\nu}{\omega} \right)^{\frac{1}{2}} = -\left(\frac{\partial \theta}{\partial \zeta} \right)_{\zeta = 0}$$
$$= -\tfrac{1}{2} \sigma^{\frac{1}{2}} \tau^{-\frac{1}{2}} \left(\frac{\partial \theta}{\partial \xi} \right)_{\xi = 0}$$

Hence from (9), (14), (15) and (16) we have

$$Nu_r = 0{\cdot}564 \, \sigma^{\frac{1}{2}} \tau^{-\frac{1}{2}} + 0{\cdot}1275 \, (m + 1) \, \tau -$$
$$- \sigma^{-\frac{1}{2}} \tau^{5/2} \left\{ {\cdot}0012 \, (17m^2 + 20m - 9) + \frac{{\cdot}0071}{\sigma} (m + 1) \right\} + 0 \, (\tau^4) \quad (17)$$

Solution for larger τ

For large values of τ, the above method will not hold, because in particular for very small values of σ, the thermal boundary layer will be much thicker than the velocity boundary layer. So we consider the energy equation in the steady state and seek perturbations of the thermal boundary layer very near the disc. Here we retain only those terms in velocity components which are linear in ζ. This is more appropriate while considering the solution for large σ. In practice this approximation is found to be acceptable when σ is of the order unity. The energy equation (4) (for steady state) takes the form

$$\frac{\partial^2 \theta}{\partial \zeta^2} - \sigma \, H \, \frac{\partial \theta}{\partial \zeta} - \sigma \, F \, m \, \theta = 0 \quad (18)$$

Let
$$\theta = \theta_o \, (\zeta) + \theta_1 \, (\zeta, \tau) \quad (19)$$

θ_o being the unperturbed value and θ_1 satisfies

$$\frac{\partial^2\theta_1}{\partial\zeta^2} - \sigma H \frac{\partial\theta_1}{\partial\zeta} - \sigma F m \theta_1 - \sigma \frac{\partial\theta_1}{\partial\tau} = 0 \tag{20}$$

with boundary conditions

$$\left.\begin{aligned} \theta_1 (0, \tau) &= 0 \\ \theta_1 (\infty, \tau) &= 0 \end{aligned}\right\} \tag{21}$$

Assuming perturbation

$$\theta_1 (\zeta,\tau) = x (\tau)\, \phi (\zeta) \tag{22}$$

(18) becomes

$$\frac{\partial^2\phi}{\partial\zeta^2} - \sigma H \frac{\partial\phi}{\partial\zeta} - \sigma F m \phi - \frac{\sigma\phi}{x} \frac{\partial x}{\partial\tau} = 0 \tag{23}$$

Retaining only ζ order terms in the velocity components

$$\frac{\partial^2\phi}{\partial\zeta^2} - \sigma a_o \zeta m \phi - \frac{\sigma\phi}{x} \frac{\partial x}{\partial\tau} = 0 \tag{24}$$

$$\frac{\partial^2\phi/\partial\zeta^2 - \sigma a_o \zeta m \phi}{\sigma\phi} = \frac{\partial x/\partial\tau}{x} = -p_n \tag{25}$$

which gives $\qquad x = \exp(-p_n \tau) \times \text{const}$

and the general solution is

$$\theta_1 = \sum_{n=1}^{\infty} C p_n \phi_{p_n} (\zeta) \exp(-p_n \tau) \tag{26}$$

Also

$$\frac{\partial^2\phi_{p_n}}{\partial\zeta^2} - m \sigma a_o \zeta \phi_{p_n} + p_n \sigma \phi_{p_n} = 0 \tag{27}$$

subject to

$$\phi_{p_n} (0) = 0 \; ; \quad \phi_{p_n} (\infty) = 0 \tag{28}$$

We define a new variable

$$s = \left(\frac{\sqrt{\sigma}}{m a_o}\right)^{\frac{2}{3}} (m a_o \zeta - p_n)$$

which transforms (27) to

$$\frac{\partial^2\phi_{p_n}}{\partial s^2} - s \phi_{p_n} = 0 \tag{29}$$

subject to conditions (28). The solution of (29) satisfying the second boundary condition in (28) is

$$\phi_{p_n} = A_i (s) \tag{30}$$

where $Ai(s)$ denotes the Airy function. For $s > 0$, $Ai(s)$ has no zeros for finite s, but for $s<0$, $Ai(s)$ has an infinite number of zeros at $s = -s_n$ with $Ai(-s_n) = 0$. The smallest value s_1, which has the largest contribution to $x(\tau)$ in (22), and for which (30) holds is $s_1=2\cdot338$ and the corresponding value of p_n which the first condition of (28) gives at $\zeta = 0$ is

$$p_1 = 2\cdot338 \left(\frac{ma_o}{\sqrt{\sigma}}\right)^{\frac{2}{3}}$$

Thus as $\tau \to \infty$ the dominant term in θ_1 is given by

$$\theta_1 = C_{p_1} \exp(-p_1 \tau)\, Ai(-s_1) \tag{31}$$

The constant C_{p_1} will depend upon the initial growth of the thermal boundary layer. The local Nusselt number as defined earlier is given by

$$Nu_r = Nu_s - 0\cdot5664\,(m\alpha)^{1/3}\, C_{p_1} \exp(-p_1 \tau) \tag{32}$$

where Nu_s is the local Nusselt number for steady state and it has been tabulated for various values of σ and m by A.A. Hayday[7]. From (33) it follows that the steady state is approached as an exponential decay.

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REFERENCES

1. VON KARMAN, T., *Z. A. M. M.*, 1 (1921), 233.
2. COCHRAN, W.G., *Proc. Camb. Phil. Soc.*, 30 (1934), 365.
3. MILLISAPS, K. & POHLHAUSEN, K., *J. Aero. Sci.*, 19 (1952), 120.
4. MORGAN, G.W. & WARNER, W.H., *J. Aero. Sci.*, 23 (1956), 937.
5. LIGHTHILL, M.J., *Proc. Roy Soc.*, A202 (1950), 369.
6. RILEY, N., *J.Fluid Mech.*, 17 Part 2 (1963), 97.
7. HAYDAY, A.A., *J. Heat Transfer, Trans. ASME*, C87 (1965), 445.