PROGRESSIVELY CENSORED SAMPLES IN LIFE TESTING WITH CHANGING FAILURE RATE

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In life testing experiments, progressively censored samples arise when at various stages of an experiment some of the surviving items are eliminated from further observations, while the test continues with the remaining items. It has been assumed here that the failure rate changes at each stage and the method of maximum likelihood to estimate the failure rate at each stage has been used.

In life test experiments, observations on the lives of items are time ordered and it is common practice to terminate the experiment when certain number of items have failed or certain stipulated time has elapsed. For such tests huge literature[1] is available. Recently, Cohen[2] introduced progressively censored samples in life testing. In such experiments, after the first stage of the experiment is over, some items fail and some are removed and the test continued with the remaining items. This process is repeated at each stage. The idea of removing some items at every stage stems from the fact that these items might be required for use somewhere else.

However, in services certain stores and equipments are subjected to regular check up even though they are functioning normally. When such items are placed on life test at some stages the items that have not failed are checked up and over-hauled, repairing the minor defects. This, naturally, changes the life time distribution of the items and consequently the failure rate also changes. It has been assumed that the times of censoring coincide with the times of regular check ups and, thus, are predetermined so that the failure rate changes at the time of censoring. Moreover, the lives of items have been assumed to follow the exponential distribution.

STAGE CASE

Two stage case

In this case the censoring occurs only at one time and the experiment is terminated as soon as certain stipulated time elapses after censoring. Let n items be placed on test, the life time of the items following the exponential distribution with failure rate λ_1. The failure rate is defined as the probability of instantaneous failure of the item at a time provided the item has not failed upto that time. If $f(t)$ represents the probability density function of the life of an item and $F(t)$ represents the distribution function then failure rate is given by[3],

$$\frac{f(t)}{1-F(t)}$$

It can be easily seen that the failure rate is constant in the case of exponential distribution.

Let n_1 be the number of items that failed by time T_1 and r_1 be the number of items that are removed after time T_1. Let the lives of remaining items follow the exponential distribution with failure rate λ_2.

Let n_2 be the number of items that failed further by time T_2, so that the remaning $n - n_1 - r_1 - n_2 = r_2$ (say) items are removed from the test and the experiment terminated. It is worth noting that n, r_1, T_1 and T_2 have been assumed to be fixed while n_1 and n_2 have been assumed to be random variables. However, if n_1 happens to be greater than $n - r_1$, then the experiment has to be terminated at T_1 itself.

To form the likelihood function for such a sample we proceed as follows :

From our assumption about failure rate being λ_1 and λ_2 in two stages we have

$$\frac{f(t)}{1 - F(t)} = \lambda_1 \qquad 0 < t \leqslant T_1 \tag{1}$$

$$\frac{f(t)}{1 - F(t)} = \lambda_2 \qquad T_1 < t \leqslant \infty \tag{2}$$

Integrating (1) from 0 to t and (2) from T_1 to t, we get

$$1 - F(t) = \begin{cases} 1 - F_1(t) = \exp. (-\lambda_1 t) \ ; & 0 < t \leqslant T_1 \\ 1 - F_2(t) = \exp. [(\lambda_2 - \lambda_1) T_1 - \lambda_2 t] \ ; & T_1 < t \leqslant \infty \end{cases}$$

and

$$f(t) = \begin{cases} f_1(t) = \lambda_1 \exp. (-\lambda_1 t) \ ; & 0 < t \leqslant T_1 \\ f_2(t) = \lambda_2 \exp. [(\lambda_2 - \lambda_1) T_1 - \lambda_2 t] \ ; & T_1 < t \leqslant \infty \end{cases}$$

Now the likelihood function $P(S)$ of the sample is given by

$$P(S) = \text{Const.} \prod_{j=1}^{n_1} f_1(t_j) \prod_{j=n_1+1}^{n_1+n_2} f_2(t_j) \, [1 - F_1(T_1)]^{r_1} [- F_2(T_2)]^{r_2}$$

where $t_1, t_2, \ldots\ldots, t_{n_1}, t_{n_1+1}, \ldots\ldots\ldots\ldots t_{n_1+n_2}$ are the observed times for failures of $n_1 + n_2$ items

$$= \text{Const.} \ \lambda_1^{n_1} \exp. \left(-\lambda_1 \sum_{j=1}^{n_1} t_j\right)$$

$$\exp. [-n_2 T_1 (\lambda_1 - \lambda_2)] \cdot \exp. \left(-\lambda_2 \sum_{j=n_1+1}^{n_1+n_2} t_j\right)$$

$$\cdot \exp. (-r_1 \lambda_1 T_1) \cdot \exp. [-r_2 T_1 (\lambda_1 - \lambda_2) - r_2 \lambda_2 T_2] \tag{3}$$

Taking the logarithm of (3) we get

$$\log \ \mathrm{P(S)} = L \ (\text{say})$$

$$= \text{Const.} + n_1 \log \lambda_1 - \lambda_1 \sum_{j=1}^{n_1} t_j$$

$$- n_2 T_1 (\lambda_1 - \lambda_2) + n_2 \log \lambda_2 - \lambda_2 \sum_{j=n_2+1}^{n_1+n_2} t_j$$

$$- r_1 \lambda_1 T_1 r_2 \lambda_2 (T_2 - T_1) - r_2 \lambda_1 T_1 \tag{4}$$

On differentiating (4) with respect to λ_1 we have

$$\frac{\partial L}{\partial \lambda_1} = \frac{n_1}{\lambda_1} - \sum_{j=1}^{n_1} t_j - n_2 T_1 - r_1 T_1 - r_2 T_1 \ . \tag{5}$$

Equating (5) to zero we get the estimate for λ_1 as

$$\hat{\lambda}_1 = \frac{n_1}{\sum_{j=1}^{n_1} t_j + (n - n_1)\, T_1}$$

Similarly finding $\frac{\partial L}{\partial \lambda_2}$ and equating it to zero we get the estimate for λ_2 :

$$\hat{\lambda}_2 = \frac{n_2}{\sum_{j=n_1+1}^{n_1+n_2} t_j + r_2\, T_2 - T_1\,(n_2 + r_2)}$$

The asymptotic variance-covariance matrix of $\hat{\lambda}_1$, $\hat{\lambda}_2$ is given as

$$\begin{bmatrix} V\,(\hat{\lambda}_1) & \text{Cov.}\,(\hat{\lambda}_1, \hat{\lambda}_2) \\ \text{Cov}\,(\hat{\lambda}_1, \hat{\lambda}_2) & V\,(\hat{\lambda}_2) \end{bmatrix} = \begin{bmatrix} E\left(-\frac{\partial^2 L}{\partial \lambda_1^2}\right) & E\left(-\frac{\partial^2 L}{\partial \lambda_1 \partial \lambda_2}\right) \\ E\left(-\frac{\partial^2 L}{\partial \lambda_1 \partial \lambda_2}\right) & E\left(-\frac{\partial^2 L}{\partial \lambda_2^2}\right) \end{bmatrix}^{-1}$$

Now it can be easily seen that

$$\frac{\partial^2 L}{\partial \lambda_1^2} = -\frac{n_1}{\lambda_1^2}$$

$$\frac{\partial^2 L}{\partial \lambda_1 \partial \lambda_2} = O$$

$$\frac{\partial^2 L}{\partial \lambda_2^2} = -\frac{n_2}{\lambda_2^2}$$

Hence

$$E\left(-\frac{\partial^2 L}{\partial \lambda_1^2}\right) = \frac{E\,(n_1)}{\lambda_1^2}$$

and

$$E\left(-\frac{\partial^2 L}{\partial \lambda_2^2}\right) = \frac{E\,(n_2)}{\lambda_2^2}$$

To find $E(n_1)$ we take the expectation of both sides of (5) and equate to zero. Thus

$$E\left(\frac{\partial L}{\partial \lambda_1}\right) = \frac{E\,(n_1)}{\lambda_1} - n\,T_1 + E\,(n_1)\,T_1 - E\,(n_1).\,E\,(t/t \leqslant T_1) = O$$

and since,

$$E\,(t\,/\,t \leqslant T_1) = \frac{\lambda_1}{1 - \exp\,(-\lambda_1 T_1)} \int_0^{T_1} t.\,\exp.\,(-\lambda_1\, t)\;dt.$$

$$= \frac{-\,T_1 \exp.\,(-\lambda_1\, T_1)}{1 - \exp.\,(-\lambda_1\, T_1)} + \frac{1}{\lambda_1}$$

we get

$$E\,(n_1) = n\,\{1 - \exp.\,(-\lambda_1\, T_1)\}$$

Similarly

$$E(n_2) = \{ n - E(n_1) - r_1 \} \{ 1 - \exp. (T_1 - T_2) \lambda_2 \}$$

$$= n \left\{ \exp. (-\lambda_1 T_1) - \frac{r_1}{n} \right\} \{ 1 - \exp. (-\lambda_2) (T_2 - T_1) \}$$

Finally

$$V(\hat{\lambda}_1) = \frac{\lambda^2_1}{n \{1 - \exp. (-\lambda_1 T_1)\}}$$

$$V(\hat{\lambda}_2) = \frac{\lambda^2_2}{n \left\{ \exp (-\lambda_1 T_1) - \frac{r_1}{n} \right\} \{1 - \exp. (-\lambda_2) (T_2 - T_1)\}}$$

$$\text{Cov.} (\hat{\lambda}_1, \hat{\lambda}_2) = O$$

It is worth noting that $V(\hat{\lambda}_2)$ does not exist if $\frac{r_1}{n} \geqslant \exp (-\lambda_1 T_1)$

and it is obvious because $\hat{\lambda}_2$ itself does not exist, because the experiment is terminated at T_1.

K stage case

In this case censoring occurs at times $T_1, T_2 \ldots\ldots, T_{k-1}$ and the experiment is finally terminated at time T_k. The failure rate also changes at times $T_1, T_2, \ldots\ldots T_{k-1}$. Assuming that the failure rate is λ_i in the interval (T_{i-1}, T_i); $i = 1, 2, \ldots\ldots, k$ where $T_0 = O$, the probability density function and distribution functions can be derived as in the case of two stages to be :

$$f(t) = \begin{cases} f_1(t) = \lambda_1 \exp. (-\lambda_{,}t) & O < t \leqslant T_1 \\ f_2(t) = \lambda_2 \exp. [-T_1 (\lambda_1 - \lambda_2) - \lambda_2 t]; & T_1 < t \leqslant T_2 \\ f_3(t) = \lambda_3 \exp. [-T_1 (\lambda_1 - \lambda_2) - T_2 (\lambda_2 - \lambda_3) - \lambda_3 t]; & T_2 < t \leqslant T_3 \\ - \quad - \quad - \quad - \quad - \quad - \quad - & - \\ - \quad - \quad - \quad - \quad - \quad - \quad - & - \\ f_k(t) = \lambda_k \exp. [-T_1 (\lambda_1 - \lambda_2) - \ldots - T_k (\lambda_{k-1} - \lambda_k) - \lambda_k t]; & T_{k-1} < t \leqslant T_k \end{cases}$$

$$F(t) = \begin{cases} F_1(t) = 1 - \exp. (-\lambda_1 t); & O < t \leqslant T_1 \\ F_2(t) = 1 - \exp. [-\lambda_1 (T_1 - T_2) - \lambda_2 t]; & T_1 < t \leqslant T_2 \\ F_3(t) = 1 - \exp. [-T_1 (\lambda_1 - \lambda_2) - T_2 (\lambda_2 - \lambda_3) - \lambda_3 t]; & T_2 < t \leqslant T_3 \\ - \quad - \quad - \quad - \quad - \quad - \quad - & - \\ - \quad - \quad - \quad - \quad - \quad - \quad - & - \\ F_k(t) = 1 - \exp. [-T_1 (\lambda_1 - \lambda_2) - \ldots - T_k (\lambda_{k-1} - \lambda_k) - \lambda_k t]; & T_{k-1} < t \leqslant T_k \end{cases}$$

Now using the same notations as in the case of two stages the likelihood function $P(S)$ of the sample is given by

$$P(S) = \text{Const.} \prod_{j=1}^{n_1} f_1(t_j) \prod_{j=n_1+1}^{n_1+n_2} f_2(t_j) \ldots\ldots \prod_{j=n_1+\ldots+n_{k-1}+1}^{n_1+\ldots+n_k} f_k(t_j)$$

$$\cdot [1-F_1(T_1)]^{r_1} [1-F_2(T_2)]^{r_2} \ldots\ldots [1-F_k(T_k)]^{k-1}$$

It has been assumed that n, r_i s and T_i s are fixed while n_i s are random variables. However, if at ith stage

$$n_i > n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i} r_j$$

then the experiment has to be terminated at the ith stage itself.

$$P(S) = \text{Const.} \; \lambda_1^{n_1} \exp.\left(-\lambda_1 \sum_{j=1}^{n_1} t_j\right) \lambda_2^{n_2} \exp\left[-n_2 T_1(\lambda_1-\lambda_2)\right.$$

$$\left. - \lambda_2 \sum_{j=n_1+1}^{n_1+n_2} t_j\right] \ldots\ldots \lambda_k^{nk} \exp.\left[-n_k T_1(\lambda_1-\lambda_2) - \ldots - n_k T_{k-1}(\lambda_{k-1}-\lambda_k)\right.$$

$$\left. - \lambda_k \sum_{j=n_1+\ldots+n_{k-1}+1}^{n_1+\ldots+n_k} t_j\right] \exp(-r_1 \lambda_1 T_1)$$

$$\cdot \exp[-r_2 T_1 (\lambda_1-\lambda_2) - r_2 \lambda_2 T_2] \ldots\ldots$$

$$\cdot \exp[-r_k T_1 (\lambda_1-\lambda_2) - r_k T_2(\lambda_2-\lambda_3) - \ldots - r_k T_{k-1}(\lambda_{k-1}-\lambda_k) - r_k \lambda_k T_k]$$

Taking the logarithm of both sides and differentiating with respect to $\lambda_1, \lambda_2 \ldots \lambda_k$ separately and equating to zero we get the estimate for λ_i ($i = 1, 2, \ldots k$) as

$$\hat{\lambda}_i = \frac{n_i}{\left[\sum_{n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i} t_j + \left\{n - \sum_{j=1}^{i-1}(n_j+r_j) - n_i\right\} T_i - \left\{n - \sum_{j=1}^{i-1}(n_j + r_j)\right\} T_{i-1}\right]}$$

and as in the case of two stages, the asymptotic variances and covariance of estimates $\hat{\lambda}_1, \hat{\lambda}_2, \ldots \hat{\lambda}_k$ can be found to be

$$V(\hat{\lambda}_i) = \frac{\lambda_i^2}{E(n_i)} \quad ; \; i = 1, 2, \ldots k$$

$$\text{Cov.}\,(\hat{\lambda}_i, \hat{\lambda}_j) = 0 \quad ; \; i \neq j \,; i, j = 1, 2, \ldots k$$

where

$$E(n_i) = \frac{\left\{n - \sum_{j=1}^{i-1} r_j - \sum_{j=1}^{i-1} E(n_j)\right\} \left\{F_i(T_i) - F_{i-1}(T_{i-1})\right\}}{1 - F_{i-1}(T_{i-1})}$$

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