DIFFRACTION OF PLANE STRAIGHT SHOCK WAVE PAST A CONVEX CORNER

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Lighthill's problem on the diffraction of shock wave past a convex corner has been investigated by the application of Ting and Ludloff's method. The numerical computation for the pressure distribution along the wall has been carried out for $P/P_\circ=3$ and $P/P_\circ=10$. The results agree with those of Lighthill.

Lighthill[1] considered the diffraction of plane straight shock wave past a small bend and gave a detailed analysis for the case of a convex corner. Ting and Ludloff[2] made a more general approach to the problem and obtained explicitly the expressions for pressure and density in the whole domain behind the advancing shock wave. In particular the pressure and density field were computed by them for a concave corner. In this note the pressure distribution behind the diffracted shock wave has been computed for a convex corner by Ting and Ludloff's method and the results have been compared with those of Lighthill for two shock strengths ($P/P_\circ=3$ and $P/P_\circ=10$). This work has been carried out to extend the case of diffraction of normal shock wave to that of oblique shock wave.

FORMULATION OF THE PROBLEM

After the shock wave has crossed the corner it gets diffracted and a non-uniform region bounded by the diffracted shock, Mach circle and the wall is formed. Let the velocity of the shock be Uo and the velocity of flow behind it be $(Uo—U)$. A coordinate system (x, y) is chosen fixed in the flow behind the shock with the x—axis coinciding with the plane wall. Since air enters the non-uniform region across a curved shock, a rotational motion is expected. Assuming the disturbance within this region to be very small, we obtain the following relations from the equations of two-dimensional unsteady rotational motion.

$$\left.\begin{aligned} \rho_t^{(1)} + Ru_x^{(1)} + Rv_y^{(1)} &= O \\ Ru_t^{(1)} &= -p_x^{(1)} \\ Rv_t^{(1)} &= -p_y^{(1)} \\ p_t^{(1)} &= C^2 \rho_t^{(1)} \end{aligned}\right\} \tag{1}$$

where $p^{(1)}$, $\rho^{(1)}$, $u^{(1)}$ and $v^{(1)}$ denote the first order perturbations in pressure, density, velocity in the x—direction and velocity in the y—direction respectively. R is the unperturbed density and C is the velocity of sound behind the shock.

Eliminating $\rho^{(1)}$, $u^{(1)}$ and $v^{(1)}$ from equations (1) we, get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)p^{(1)} = 0 \tag{2}$$

The problem now reduces to finding the solution of the equation (2). It is then necessary to specify the initial and boundary conditions for $p^{(1)}$.

BOUNDARY CONDITIONS FOR $p^{(1)}$ IN THE NON-UNIFORM REGION

(*i*) *On the incident shock front*

The disturbed shock front (Fig. 1) can be expressed by the equation

$$x = Ut - \epsilon\, \psi^{(1)}(y, t) + O(\epsilon^2) \tag{3}$$

Then the shock inclination is given by

$$\theta = \frac{\partial x}{\partial y} = -\epsilon\, \psi_y^{(1)}(y, t) + O(\epsilon^2) \tag{4}$$

Shock velocity in the x—direction

$$U_s = \frac{dx}{dt} = \frac{\partial x}{\partial t} + O(\epsilon^2) = U - \epsilon\, \psi_t^{(1)}(y, t) + O(\epsilon^2) \tag{5}$$

and in the y—direction

$$V_s = U_s \tan\theta = -\epsilon\, \psi_y^{(1)}(y, t) + O(\epsilon^2) \tag{6}$$

Let us define the various components of velocity of flow relative to the shock as follows:

q_n = normal component of velocity (behind)

$$= -U + \epsilon\, \psi_t^{(1)} + \epsilon\, u^{(1)} + O(\epsilon^2) \tag{7}$$

q_τ = tangential component of velocity (behind)

$$= \epsilon\, v^{(1)} + O(\epsilon^2) \tag{8}$$

q_{n_o} = normal component of velocity (in front)

$$= -U_o + \epsilon\, \psi_t^{(1)} + O(\epsilon^2) \tag{9}$$

q_{τ_o} = tangential component of velocity (in front)

$$= (U_o - U),\ \epsilon\, \psi_y^{(1)} + O(\epsilon^2) \tag{10}$$

Substituting the values of q_n, q_{no}, q_T and q_{T_o} in the oblique shock conditions on the curved shock front ($x = Ut$, $y > 0$, t), we obtain

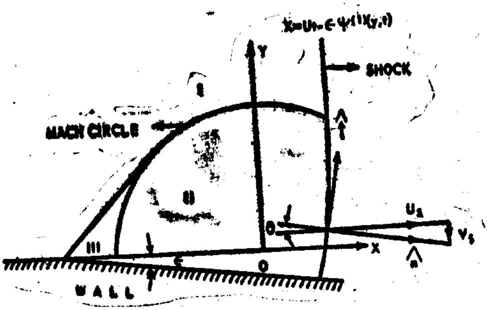

Fig. 1—Geometry of curved shock.

$$\left.\begin{aligned} C^2 \rho^{(1)} &= (1 + \Omega_o)\, p^{(1)} \\ u^{(1)} &= \frac{\Omega_1}{RC}\, p^{(1)} \\ (U_o - U)\, \psi_t^{(1)} &= \frac{\Omega_2}{R}\, p^{(1)} \\ (U_o - U)\, \psi_y^{(1)} &= v^{(1)} \end{aligned}\right\} \tag{11}$$

where

$$\Omega_0 = -\frac{(\gamma-1)\,(M^2-1)^2}{[M^2\,(\gamma-1)+2]\,M^2}, \qquad M = \frac{U}{C}$$

$$\Omega_1 = \tfrac{1}{2}M\,\frac{(3\,\gamma-1)\,M^2+(3-\gamma)}{M^2\,(\gamma-1)+2},$$

and

$$\Omega_2 = -\frac{1-M^2}{M^2}$$

Eliminating $u^{(1)}$, $v^{(1)}$, $\psi^{(1)}$ and $\rho^{(1)}$ from the equations (11) by using the equation (1), we get the following boundary conditions for $p^{(1)}$ on the shock front

$$D_{x,t}\; p^{(1)}(x = Ut,\, y > 0,\, t) = 0 \tag{12}$$

where

$$D_{x,t} = (\Omega_1 + M + M\,\Omega_2)\,\frac{\partial^2}{\partial t^2} + (1 + M^2 + 2\,M\,\Omega_1)\,C\,\frac{\partial^2}{\partial x\,\partial t} + (M^2\,\Omega_1 + M - M\,\Omega_2)\,C^2\,\frac{\partial^2}{\partial x^2}$$

(*ii*) *On the wall*

A first approximation is

$$v^{(1)}(x \leqslant Ut,\, y = 0,\, t) = -(U_o - U)\,f'\,[x + (U_o - U)\,t] \tag{13}$$

where the prime means differentiation with respect to the whole argument.

From equation (1) the boundary condition for $p^{(1)}$ on the wall $y=0$ is

$$p^{(1)}_y\,(x < U\,t,\, y = 0,\, t) = R\,(U_o - U)^2\,f''\,[x + (U_o - U)\,t] \tag{14}$$

(*iii*) *On the Mach circle*

As $p^{(1)}$ varies continuously across the Mach circle the boundary condition may simply be prescribed as $p^{(1)} \longrightarrow 0$ *as* $\sqrt{x^2+y^2} \longrightarrow \infty$ (15)

INITIAL CONDITIONS FOR $p^{(1)}$

The two initial conditions are

$$\left.\begin{aligned} &p^{(1)}\,(x \leqslant U\,t,\, y \geqslant 0,\, t \leqslant 0) = 0) \\ \text{and}\quad &p_t^{(1)}(x \leqslant U\,t,\, y \geqslant 0,\, t \leqslant 0) = 0) \end{aligned}\right\} \tag{16}$$

DERIVATION OF THE ANALYTIC SOLUTION

By using Lorentz transformation

$$\left.\begin{aligned} &\bar{x} = (x - Ut)\,/\,\sqrt{1-M^2}\,, \quad \bar{y} = y \\ \text{and}\quad &\bar{t} = (Ct - M\,x)/\sqrt{1-M^2} \end{aligned}\right\} \tag{17}$$

the wave equation becomes

$$p^{(1)}_{\bar{x}\,\bar{x}} + p^{(1)}_{\bar{y}\,\bar{y}} - p^{(1)}_{\bar{t}\,\bar{t}} = 0 \tag{18}$$

and the boundary conditions become

$$p^{(1)} \rightarrow 0 \text{ as } \sqrt{\bar{x}^2 + \bar{y}^2} \rightarrow \infty , \tag{19}$$

$$p_y^{(1)} (\bar{x} < 0, \bar{y} = 0, \bar{t}) = - R\, C^2 A_o\, f''[\bar{a} (\bar{\lambda}_o \bar{x} + t)] \tag{20}$$

$$D_{\bar{x}, \bar{t}}\; p^{(1)} (\bar{x} = 0, \bar{y} > 0, \bar{t}) = 0 \tag{21}$$

where

$$\bar{a} = \frac{U_o}{C \sqrt{1 - M^2}}, \quad \bar{\lambda}_o = \left(1 - M^2 + M \frac{U_o}{C}\right) \frac{C}{U_o},$$

$$A_o = - \frac{(U_o - U)^2}{C^2} \text{ and } D_{\bar{x}, \bar{t}} = \frac{1}{M_o^2} \frac{\partial^2}{\partial \bar{t}^2} + 2 M \frac{\partial^2}{\partial \bar{x}\, \partial \bar{t}} + \frac{\partial^2}{\partial \bar{x}^2},$$

$$\left(M_o \text{ being } \frac{U_o}{C_o}\right)$$

The initial conditions are now

$$p^{(1)} = p_{\bar{t}}^{(1)} = 0 \; (for \; \bar{t} \leqslant 0) \tag{22}$$

Like Ting and Ludloff[2] these boundary and initial value problems can be solved with the help of Possio Integral.

$$p^{(1)} \left(\bar{x}, \bar{y}, \bar{t} \right) = - \frac{1}{\pi} \iint \frac{p_y^{(1)} (\xi, O, T)}{\sqrt{(\bar{t} - \tau)^2 - (\bar{x} - \xi)^2 - \bar{y}^2}}\, d \xi\, d_T \tag{23}$$

where $p_y^{(1)}$ denotes the strength of the temporary sources of disturbance and the denominator represents the pseudo distance between source point ξ, O, T, and x, y, t. Therefore from (23), inserting proper integration limits, we obtain

$$p^{(1)} = \frac{R\, C^2 A_o}{\pi} \int_0^{\bar{t} - \bar{y}} dt \int_{\bar{x} - \sqrt{(\bar{t} - \tau)^2 - \bar{y}^2}}^{0} \frac{f'' [\bar{a} (\tau + \bar{\lambda}_o \xi)]}{\sqrt{(\bar{t} - \tau)^2 - (\bar{x} - \xi)^2 - \bar{y}^2}}\, d\xi$$

$$+ \sum_{i=1}^{2,3} \frac{RC^2 Ai}{\pi} \int_0^{\bar{t} - \sqrt{\bar{x}^2 + \bar{y}^2}} d\xi \frac{f'' [\bar{a} (\tau - \bar{\lambda}_i \xi)]}{\sqrt{(\bar{t} - \tau)^2 - (\bar{x} - \xi)^2 - \bar{y}^2}} \tag{24}$$

where $\bar{\lambda}_1 = \bar{\lambda}_0$

$$A_1 = - A_0 \frac{\bar{\lambda}_0^2 + 2 M \bar{\lambda}_o + (1/M_0^2)}{\bar{\lambda}_0^2 - 2 M \lambda_0 + (1/M_0^2)}$$

$\bar{\lambda}_2$ and $\bar{\lambda}_3$ are the roots of the quadratic equation

$$\bar{\lambda}^2 - 2 M \bar{\lambda} + (1 / M_0^2) = 0$$

and A_2 and A_3 are the solutions of the following simultaneous equations

$$A_2 + A_3 + A_1 + A_0 = -\frac{8}{\gamma+1}\,\frac{MU_0}{C}$$

and

$$\bar{\lambda}_2 A_2 + \bar{\lambda}_3 A_3 + \bar{\lambda}_1 A_1 + \bar{\lambda}_0 A_0 = -2M\left(\frac{8}{\gamma+1}\,\frac{M\,U_0}{C} + 2A_0\right)$$

In the case when the angle is very small, say ϵ, the equation of the wall surface beyond the corner is $y = -\epsilon f(x')$

where $$x' = x + (U_0 - U)t.$$

In the simplest case $f(x') = x'$

Then $f'(x') = 0$ for $x' > 0$ and $f'(x') = 0$ for $x' < 0$. Furthermore,

$$\left.\begin{aligned} & f''(x') = 0 \qquad (\text{for } x' \neq 0) \\ & \int_{0-}^{0+} f''(x')\,dx' = 1 \end{aligned}\right\} \tag{25}$$

Using equation (25) the pressure integral (24) yields

$$\overset{(1)}{p}(\sigma, n) = \frac{RC^2}{\pi U_0/C}\sum_{i=0}^{3}\frac{1-\lambda_i M}{(1-\lambda_i^2)^{\frac{1}{2}}}\,A_i\,Cos^{-1}\frac{[\lambda_i - \sigma]}{[(1-\lambda_i\,\sigma)^2 + (\lambda^2_i - 1)\eta^2]^{1/2}}.$$

$$(\text{for } \lambda_i < 1) \tag{26}$$

where

$$\lambda_o = \frac{M - \bar{\lambda}_o}{1 - M\bar{\lambda}_o},\quad \bar{\lambda}_i = \frac{\bar{\lambda}_i + M}{1 + \bar{\lambda}_i M},\ (i = 1, 2, 3,),\ \sigma = \frac{x}{Ct} \text{ and } \eta = \frac{y}{Ct}$$

In the event that $\lambda_i > 1$, Cos^{-1} terms will be replaced by corresponding $Cosh^{-1}$ terms

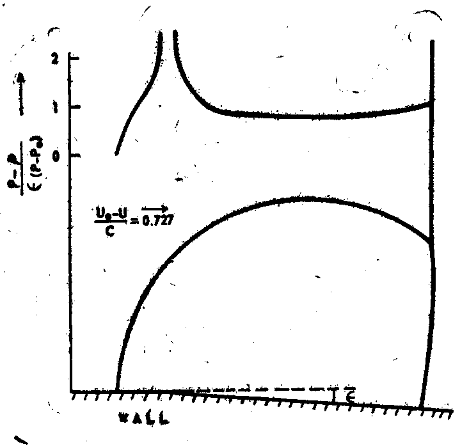

Fig. 2—Wall pressure distribution and shape of distribution region for $p/p_0 = 3$

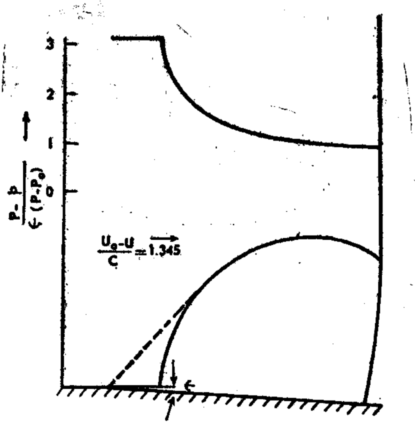

Fig. 3—Wall pressure distribution and shape of distribution region for $p/p_0 = 10$

Using this relation the pressure distribution along the wall has been computed for $\frac{P}{P_\circ} = 3$ (Fig. 2) and $\frac{P}{P_\circ} = 10$ (Fig. 3). It has been found to agree with the result obtained by Lighthill.

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