APPLICATION OF LANCHESTER'S EQUATIONS TO A TACTICAL PROBLEM

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By applying Lanchester's equations, it has been found that under certain conditions it is preferable to give an opportunity to the enemy entrenched in a well defended locality to escape rather than force him to fight by demanding his unconditional surrender.

In modern wars emphasis has been laid on unconditional surrender of the enemy. This, in turn, has led to the concept of "fight to the finish". The tactical doctrine advocated by some ancient authorities on warfare such as Sunzu (500 B.C.) and Napolean (1800 A.D.) viz, permitting an enemy, located in a well defended locality, an avenue of escape, has been ignored. This paper attempts to evaluate the 'worth' of this doctrine with the help of Lanchester's Equations.

THE PROBLEM

Consider a battle between Reds and Blues; the Blues withdrawing with the Reds in pusuit. Let θ_1 be a fraction of the Blues left behind in a well defended locality either intentionally to delay the advance of the Reds or because of the inability of the Blues to withdraw in time. Let the defences of the locality be such as to give a local superiority to the Blues so that any attempt on the part of the Reds to reduce the locality by forcing the Blues to fight results in heavy casualties to the attacking force. In order to reduce the defended locality, the Red Commander can deploy his forces in any one of the following three ways:

(*i*) deploy all his forces against the defended locality thus allowing the bulk of the Blues to escape and re-group and possibly receive reinforcements.

(*ii*) deploy a fraction θ of his forces against the defended locality and the remaining $(1-\theta)$ of his forces in pursuing the rest of the Blues.

(*iii*) exert the minimum pressure required against the locality to give an impression to the enemy that he would be attacked and at the same time give him an opportunity to escape and join his main body.

Course (*i*) being a particular case of course (*ii*) when $\theta=1$, courses (*ii*) and (*iii*) only will be considered.

In deciding the course of action to be adopted by the Red Commander, the losses, which he is likely to sustain in the entire course of battle with the Blues, have to be estimated and the course which is likely to minimise his losses is to be recommended. It is assumed that $(1-\theta)R_0$ Reds are in pursuit of the Blues outside the locality and do not give them a battle until the battle for the locality is over.

NOTATIONS

R_0 =Initial strength of the Reds,

B_0 =Initial strength of the Blues,

θ_1 = Fraction of the Blues left behind in the locality,

θ = Fraction of the Reds deployed against the locality,

R_1 = The number of the Reds left after reduction of the defended locality,

R_2 = The number of the Reds left after the entire battle,

b_1 = The number of reinforcements the Blues Commander is likely to receive if course (*i*) is adopted,

R_3 = The number of the Reds remaining if course (*iii*) is adopted,

E_1 = The exchange rate when the defended locality is attacked,

E = The exchange rate outside the defended locality, and

k = The fraction of initial strength to which the enemy forces are reduced to end the battle,

ANALYSIS

If R_o and B_o denote the initial number of Red and Blue forces and r and b are the corresponding numbers of Red and Blue forces left over at time 't' after the commencement of the battle, the generalised Lanchester's equations describing the reduction in numbers on each side are:

$$dr/dt = P - Ar - Cb \tag{1}$$

$$db/dt = Q - Bb - Dr \tag{2}$$

where P and Q represent the production or replenishment rates, A and B the operational attrition rates and C and D the effectiveness or efficiency of the combating troops for Red and Blue forces respectively.

Consider first the simple case of the "Square Law". If we assume that the duration of the engagement is short, and operational losses will be nil on both sides, then the first two terms of equations (1) and (2) disappear, yielding

$$dr/dt = -Cb = -b/(1+E) \tag{3}$$

$$db/dt = -Dr = -Er/(1+E) \tag{4}$$

the coefficients C, D are expressed in terms of the exchange rate, E defined as

$$E = \frac{\text{average number of Blue combatants lost}}{\text{average number of Red combatants lost}}$$

Eliminating 't' between (3) and (4) we get Lanchester's Square Law, namely

$$B_o^2 - b^2 = E\,(R_o^2 - r^2) \tag{5}$$

which can be rewritten as

$$r^2 = R_o^2 - (B_o^2 - b^2)/E \tag{6}$$

giving the number of Red forces remaining at time 't' in terms of R_o, B_o, b and E. From (6) the value of r can be obtained for a given value of b. In particular, we are interested in the value of r when the Blues surrender. This value of r is given by putting $b=kB_o$ in (6), *i.e.*

$$r^2 = R_o^2 - (1 - k^2)\,B_o^2/E \tag{7}$$

As already stated, the decision of the Red Commander to choose one of the courses of action (*i*), (*ii*) or (*iii*) depends upon the values of r corresponding to each of these courses of action. The application of (7) to the situations corresponding to the e courses yields the following results :—

Course of Action (*ii*)

(*a*) *Battle for the Pocket*—When $\theta_1 B_o$ are engaged by θR_o and the battle is fought till the Blues surrender, the numebr of Red forces remaining at the end of the battle is

$$R_1^2 = \theta^2 R_o^2 - (1 - k^2)\, \theta_1^2 B_o^2/E_1 \tag{8}$$

(*b*) *Battle outside the Pocket*—Now, this R_1 joins the other part $(1-\theta) R_o$ and engages the remaining $(1-\theta_1) B_o$ Blues. Again, the number of Red forces remaining at the end of the battle is

$$R_2^2 = \{R_1 + (1-\theta) R_o\}^2 \;\; (1-k^2)(1-\theta_1)^2 B_o^2/E \tag{9}$$

$$\begin{aligned} &= \{R_o^2 - (1-k^2) B_o^2/E\} + 2R_o^2 (\theta^2 - \theta) \\ &\quad + 2 R_o (1-\theta) \{\theta^2 R_o^2 - (1-k^2)\, \theta^2_1 B_o^2/E_1\}^{\frac{1}{2}} \\ &\quad - (1-k^2)\{\theta_1^2/E_1 + (\theta_1^2 - 2\,\theta_1)/E\} B_o^2 \end{aligned} \tag{10}$$

Course of Action (*iii*)

Considering the battle between R_o and B_o fought till the Blues surrender, the number of Red forces remaining at the time Blues have surrendered is

$$R_3^2 = R_o^2 - (1-k^2)\, B^2_o/E \tag{11}$$

By substituting (11) in (10), we have

$$\begin{aligned} R_2^2 &= R_3^2 + 2 R_o^2 (\theta^2 - \theta) + 2 R_o (1-\theta) \{\theta^2 R_o^2 - (1-k^2)\, \theta_1^2 \; B_o^2/E_1\}^{\frac{1}{2}} \\ &\quad - (1-k^2) \{\theta_1^2/E_1 + (\theta_1^2 - 2\,\theta_1)/E\} B_o^2 \\ &= R_3^2 + G, \text{ (say)} \end{aligned} \tag{12}$$

Now, (*ii*) will be preferred to (*iii*) when $R_2 > R_3$

and (*iii*) will be preferred to (*ii*) when $R_2 < R_3$.

$$R_2 \gtrless R_3 \text{ when } R_2^2 \gtrless R^2_3$$

i.e., according as $G \gtrless 0$ (13)

For given values of R_o, B_o, θ_1, θ, E, E_1 and k, this inequality can be tested and the appropriate course of action chosen.

PARTICULAR CASE

If course of action(*i*) is chosen and no reinforcements are allowed, by putting $\theta=1$ in the expression for G, the Red Commander prefers (*i*) to (*iii*) or otherwise according as

$$\left\{\theta_1^2/E_1 + (\theta_1^2 - 2\theta_1)/E\right\} B^2_o \gtrless 0 \tag{14}$$

On the other hand, if we allow for reinforcements b_1 to the Blues during the time the Reds are engaged by the Blues in the locality, a little effort will show that (*i*) will be preferred to (*iii*) or (*iii*) to (*i*) according as

$$2\theta_1 B_o (B_o + b_1) \gtrless \theta_1^2 B_o^2 (1 + E/E_1) + b_1 (b_1 + 2 B_o) \tag{15}$$

NUMRICAL ILLUSTRATION

Equation (12) gives R_2 as a function of θ, θ_1, E_1 and k. From (12) we can know the range of values of θ, for a given set of values of θ_1, E, E_1 and k, for which course of action (*ii*) is preferable to course of action (*iii*) and *vice versa*. As a numerical illustration let us take $R_o=10$, $B_o=8$, $E=1\cdot0$ and $k=0\cdot3$ so that the only parameters we shall vary are θ, θ_1, and E_1. For various sets of values θ, θ_1, E_1, the values of R_2 have been calculated and shown in Table 1. A study of the table shows that for certain combinations of E_1 and θ_1, (*iii*) is always preferable to (*ii*), for example for the combination $\theta_1=0\cdot3$ and $E_1=0\cdot1$. But for other combinations (*ii*) is preferable to (*iii*) for certain values of θ and (*iii*) to (*ii*) for certain other values of θ. For example, for the combinations $\theta_1=0\cdot2$ and $E_1=0\cdot2$ (*iii*) is preferable to (*ii*) for values of θ upto $0\cdot57$ or so while (*ii*) is preferable to (*iii*) above this value.

TABLE 1

VALUES OF R_2 FOR VARIOUS SETS OF VALUES OF θ, θ_1, & E_1

θ	θ_1 — 0·1 E_1 — 0·2	0·1 0·1	0·2 0·2	0·2 0·1	0·3 0·2	0·3 0·1	0·4 0·2	0·4 0·1
0·1	—	—	—	—	—	—	—	—
0·2	5·88	—	—	—	—	—	—	—
0·3	6·52	5·47	—	—	—	—	—	—
0·4	6·74	6·11	5·31	—	—	—	—	—
0·5	6·85	6·39	6·13	1·49	—	—	—	—
0·6	6·92	6·55	6·52	4·46	4·72	—	—	—
0·7	6·98	6·67	6·76	5·27	5·65	—	—	—
0·8	7·01	6·75	6·93	5·74	6·15	0·81	4·14	—
0·9	7·04	6·81	7·05	6·05	6·49	3·43	5·12	—
0·95	7·05	6·83	7·10	6·17	6·62	3·96	5·44	—

R_2=Number of red forces left over when course of action (*ii*) is chosen

$$= \left[R^2{}_3 + G \right]^{\frac{1}{2}}$$

R_3 =Number of red forces left over when course of action (*iii*) is chosen

$$= \left[R_o^2 - \frac{1}{E} (1 - k^2) \; B_o^2 \right]^{\frac{1}{2}}$$

$$= 6\cdot46.$$

Going back to the general case of equations (1) and (2), eliminating the time variable 't' between the two equations we get,

$$dr/db = (P - Ar - Cb)/(Q - Bb - Dr) \tag{16}$$

The solution of the above differential equation is given by

$$\left\{D\bar{R}^2-(A-B)\ \bar{R}\bar{B}-C\bar{B}^2\ \right\}^{\frac{1}{2}}\cdot\left\{\frac{2D\bar{R}+(B-A-q^{\frac{1}{2}})\bar{B}}{2\,D\bar{R}+(B-A+q^{\frac{1}{2}})\ \bar{B}}\right\}^{\frac{(A+B)}{2\,q^{\frac{1}{2}}}}=K_{\circ}\quad(17)$$

where, $\bar{R}=r-(PB-QC)/(AB-CD)$,

$\bar{B}=b-(AQ-PD)/(AB-CD)$,

$q=(A-B)^2+4\,CD$

and $K_{\circ}$ is the constant of integration to be obtained from the initial conditions $r{=}r_{\circ}$ and $b{=}b_{\circ}$. If we assume that the production rates for both sides are zero, *i.e.* $P=Q=0$ then (17) becomes

$$\left\{\ Dr^2-(A-B)\,r\,b-C\,b^2\ \right\}^{\frac{1}{2}}\left\{\frac{2\ Dr+(\ B-A-q^{\frac{1}{2}}\ )b}{2\ Dr+(\ B-A+q^{\frac{1}{2}}\ )b}\right\}^{\frac{(A+B)}{2\ q^{\frac{1}{2}}}}=K_{\circ}^{1}\quad(18)$$

Further if $A{=}B{=}\lambda$, *i.e.* if we assume that the operational attrition rates are the same on both sides which may be justified on the grounds that both the sides are fighting on the same type of terrain and the duration of the engagements is relatively short, (18) further simplifies to

$$(Dr^2-Cb^2)^{\frac{1}{2}}\cdot\left[\left\{Dr-b\,(CD)^{\frac{1}{2}}\right\}\Big/\left\{(Dr+b\,(CD)^{\frac{1}{2}}\right\}\right]^{\lambda/2\,(CD)^{\frac{1}{2}}}$$

$$=(Dr_{\circ}^{\,2}-C\,b_{\circ}^{\,2})^{\frac{1}{2}}\cdot\left[\left\{Dr_{\circ}-(CD)^{\frac{1}{2}}b_{\circ}\right\}\Big/\left\{Dr_{\circ}+b_{\circ}\,(CD)^{\frac{1}{2}}\right\}\right]^{\lambda/2\,(CD)^{\frac{1}{2}}}\qquad(19)$$

As we did earlier, we can express C and D in terms of the exhchange rate E and put (19) as

$$(Er^2-b^2)^{\frac{1}{2}}\cdot\left[\left\{E^{\frac{1}{2}}\,.\,r-b\right\}\Big/\left\{E^{\frac{1}{2}}\,.\,r+b\right\}\right]^{\lambda\,(1+E)/2E^{\frac{1}{2}}}$$

$$=(Er_{\circ}^{\,2}-b_{\circ}^{\,2}.)^{\frac{1}{2}}\left[\left\{E^{\frac{1}{2}}\,.\,r_{\circ}-b_{\circ}\right\}\Big/\left\{E^{\frac{1}{2}}\,r_{\circ}+b_{\circ}\right\}\right]^{\gamma\,(1+E)\ /2E^{\frac{1}{2}}}\qquad(20)$$

when $\lambda{=}0$, this reduces to Lanchester's square law.

None of the above solutions (17) to (20) exhibit an explicit relationship between r and b as compared to (6) in the case of the square law. Because of this fact, we cannot obtain R_1, R_2, and R_3 explicity in terms of the various prameters and $r_{\circ}$ and $b_{\circ}$ and hence cannot derive a single inequality of the type of (13) which can form the criterion for choosing between the possible courses of action (*i*), (*ii*) and (*iii*). However, for given combination of the various parameters and $r_{\circ}$ and $b_{\circ}$, R_1, R_2 and R_3 can be obtained by solving the appropriate equation for r numerically and then comparing R_2 and R_3. The method is illustrated thus :—

R_1, being the remainder of the Red forces attacking the locality when course of action (*ii*) is chosen, is obtained by solving for r one of the equations (17) to (20), say (20), when $r_o=\theta R_o$, $b_o=\theta B_o$, $E=E_1$ and $b=k\theta_1 B_o$. And R_2, being the final remainder of the Red forces when (*ii*) has been chosen, is obtained by solving for r from (20), when $r_o = (1-\theta) R_o+R_1$, $b_o= (1-\theta_1) B_o$ and $b=k(1-\theta_1)B_o$ Similarly R_3 is given by the solution of (20) when $r_o=R_o$, $b_o=B_o$ and $b=kB_o$.

We can numerically solve for R_1, R_2 and R_3 from (20) under the above mentioned conditions for various parameter combinations of θ, θ_1, E, E_1, λ, and initial strengths R_o and B_o. Assuming $R_o=10$, $B_o=8$, $E=1\cdot0$, as in the previous example, R_2 has been computed on an IBM 1620 digital computer for various combinations of θ, θ_1, E_1 and $\lambda=1\cdot0$ and is shown in Tables 2—4. For $E=1$, $\lambda=1\cdot0$, R_3 is also shown for purposes of comparison. The blank spaces indicate the cases in which either of the terms $\left\{E^{\frac{1}{2}}\, r_o - b_o\right\}$ or $\left\{(E^{\frac{1}{2}}\, r)-b\right\}$ become negative. The tables show that for the case when $\lambda=1\cdot0$, there are no combinations of θ, θ_1 which would justify the course of action (*ii*) being chosen in preference to (*iii*). Compared to the example in the case of the square law, in this more general case when $\lambda=1\cdot0$, for the values of E_1 considered, (*iii*) is always preferable to (*ii*).

The above analysis shows that using Lanchester's equations it is possible to establish a criterion by which a suitable tactic can be chosen. Also we have been able to establish the conditions under which, an enemy left over in a locality can be allowed to escape.

We have considered the problem of choosing a suitable tactic from the point of view of the Red Commander.

TABLE 2

VALUES OF R_2 FOR VARIOUS COMBINATION OF θ, θ_1, when $E_1 = 0\cdot1$

θ	θ_1 = 0·1	0·2	0·3	0·4
0·1	—	—	—	—
0·2	—	—	—	—
0·3	2·71	—	—	—
0·4	2·43	—	—	—
0·5	2·19	—	—	—
0·6	—	—	—	—
0·7	—	—	—	—
0·8	—	—	—	—
0·9	—	—	—	—

$R_3 = 3\cdot8$

TABLE 3

VALUES OF R_2 FOR VARIOUS COMBINATION OF θ, θ_1 when $E_1 = 0 \cdot 2$

θ	θ_1			
	0·1	0·2	0·3	0·4
0·1	—	—	—	—
0·2	3·18	—	—	—
0·3	2·93	—	—	—
0·4	2·76	2·56	—	—
0·5	2·63	2·30	—	—
0·6	2·53	2·08	—	—
0·7	2·45	1·94	—	—
0·8	2·38	1·69	—	—
0·9	2·33	—	—	—

$R_3 = 3 \cdot 8$

TABLE 4

VALUES OF R_2 FOR VARIOUS COMBINATION OF θ, θ_1 when $E_1 = 0 \cdot 5$

θ	θ_1			
	0·1	0·2	0·3	0·4
0·1	—	—	—	—
0·2	—	—	—	—
0·3	3·29	3·18	—	—
0·4	3·22	3·00	2·98	—
0·5	3·17	2·88	2·77	2·78
0·6	3·14	2·79	2·61	2·55
0·7	3·11	2·70	2·47	2·37
0·8	3·09	2·64	2·36	2·21
0·9	3·07	2·59	2·27	2·07

$R_3 = 3 \cdot 8$

But the problem can also be considered from the point of view of the Blue Commander. In a situation as described in the second paragraph, the Blue Commander will be faced with the question of splitting his forces, deploy a part of them in a well defended locality so as to check the advance of the Reds and contain them for sometime so that he can regroup the remainder of his forces and possibly receive reinforcments in the meantime. If he so desires, then what should be the proportion in which he should split his forces? This depends on the effectiveness of the defended locality, and estimate of the possible action the Red Commander would take, the time the Blue Commander wants to receive reinforcements, regroup and redeploy the rest of his forces and other factors like morale, luck, etc. The same analysis using Lanchester's equations can be used to determine the proportion in which the Blue Commander should split his forces to achieve the desired result.

Lanchester's equations have been used in war games to simulate the dynamics of a battle. Though Lanchester's equations do not take into consideration the effect of various other factors like weather, morale, luck, etc., (which, of course, are not subject to quantitative analysis) they still describe the dynamics of a battle to a suitable degree of accuracy as can be seen in Engel[1] and Varma[2]. The usefulness of the present analysis lies in incorporating it into a suitable war-game.

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REFERENCES

1. ENGEL, J. H., *J. Opns. Res. Soc. Amer.*, 2 (1954), 163.
2. VARMA, R. S., *Def. Sci. J.*, 7 (1957), 1.