AN APPROXIMATE SOLUTION OF TWO-DIMENSIONAL STEADY-STATE HEAT TRANSFER PROBLEMS WITH CONVECTIVE HEATING AT A CIRCULAR BOUNDARY

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Schofield's method of added thickness for solution of steady-state two-dimensional heat transfer problems with Newton heat transfer at a plane boundary has been extended to the case where such heat transfer takes place at a circular boundary. The technique has been used to study heat transfer from a buried pipe to its surrounding, the pipe axis being parallel to the ground surface. The same problem has been solved by relaxation method and numerical results have been compared graphically.

The problem of heat transfer to the ground from buried cylindrical heat sources *e.g.* cables, pipes etc. has been studied by various authors. Method of conformal transformation has been used[1] to transform the region between the buried object and the ground surface into the upper half plane to study the isothermal problem, where the temperature at the ground surface and the heat source boundary has been considered to be constant. Awbery[2] gave an exact method for treating the cases where isothermal wall condition has been assumed at one surface and Newton transfer (the heat flux being proportional to the difference of temperature) at the other. His method could be applied only to cases where Newton transfer boundary is a plane. Schofield[1] give an approximate method (method of added thickness) for treating two-dimensional steady-state heat transfer problems where the Newton transfer surface is plane. The results obtained by him compared favourably with those obtained by Awbery. The method consisted in finding a fictitious thickness to be added to the plane surface where Newton transfer is assumed such that the actual heat loss from the surface is equal to the heat loss from the isothermal fictitious surface (obtained after adding the fictitious thickness to Newton transfer surface). The problem is thus reduced to an isothermal problem which is solved by transforming the new region into the upper half plane.

It is shown in this paper that Schofield's method of added-thickness can be easily extended to the case where the Newton transfer takes place at a circular boundary. In this case we assume that the flow lines are radial within the fictitious thickness (added thickness) and this has to be obtained from the considerations that for unit time and unit length of the heat source the total heat crossing the real boundary (with Newton transfer) is equal to the total heat crossing the fictitious isothermal boundary. By the very nature of the above definition of the added thickness, it should be clear that it is a property related to the total heat at the circular boundary and not a local property and is, therefore, uniform throughout the circular boundary where the Newton transfer takes place. With this assumption it is proved that the expression for the added thickness is the same as in the plane case.

The temperature distribution in the ground above an infinitely long buried pipe losing heat by Newton transfer at its boundary has been first obtained by the 'added thickness' method. The same problem has been solved numerically by 'relaxation method' and the results have been compared graphically. The close agreement between the two results seems to justify the method of added thickness in case of Newton Transfer at a circular boundary.

THE PROBLEM

Let us consider an infinitely long buried pipe of outer radius r_1, and surface temperature V_o with its axis parallel to the ground surface, where heat is transferred to the ground by Newton Transfer at the surface of the pipe and find the steady state temperature distribution in the ground above the pipe. Let $V(r, \theta)$ denote the temperature of the ground, then V statisfies the equation.

$$\partial^2 V/\partial r^2 + (1/r)\cdot\partial V/\partial r + (1/r^2).\partial^2 V/\partial\theta^2 = 0, \qquad r > r_1 \tag{1}$$

subject to the following boundary conditions

$$V = 0 \tag{2}$$

along the ground surface and also for $r \to \infty$

and

$$-K.\ \partial V/\partial r = H\ (V_o - V), \qquad r = r_1 \tag{3}$$

where K and H are the conductivity of the ground and the surface conductivity of the material of the pipe respectively.

CALCULATION OF ADDED THICKNESS

In Fig. 1, O is the centre of the pipe, r_1 is the radius of the actual boundary B (where Newton Transfer takes place) and r_2 is the radius of the fictitious isothermal boundary B' so that $(r_1 - r_2)$, *i.e.* m is the so called added thickness or fictitious thickness.

m has to be obtained from the consideration that for unit time and unit length of the heat source the total heat crossing the real boundary B (with Newton Transfer) is equal to the total heat crossing the fictitious boundary B' kept at temperature V_o.

Let us assume that throughout the extended medium the heat flow ϕ and the temperature V are represented by the relation

$$\phi + iKV = \psi(r, \theta) \tag{4}$$

where ψ is an analytic function.

Then

$$K.\partial V/\partial r = -\ {}^1/r.\ \partial\phi/\partial\theta \tag{5}$$

The total heat that would cross any isothermal in the medium by Newton Transfer is given by

$$\xi_r = H \int_0^{2\pi} (V_o - V)\ r_1.\ d\theta \tag{6}$$

Since the limits of integration are constants,

$$d\xi_r\ /dr = -\ H\ r_1 \int_0^{2\pi} \partial V/\partial r.\partial\theta \tag{7}$$

$$= H/K. \int_0^{2\pi} \partial\phi = H/K.\ \Phi \tag{8}$$

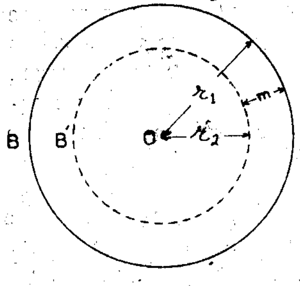

FIG. 1—Added thickness

where Φ is the total heat conducted through the boundary B per unit time per unit length of the pipe.

Integrating with respect to r between the limits r_2 and r_1 we have

$$\xi_{r_1} = H/K \ . \ \Phi \ (r_1 - r_2) \tag{9}$$

since ϕ is independent of r and ξ_{r_2} is zero. Equating ξ_{r_1} to the total heat Φ conducted through the boundary B, we get

$$m = r_1 - r_2 = K/H \tag{10}$$

SOLUTION OF THE CORRESPONDING ISOTHERMAL PROBLEM

From the above consideration it is seen that a problem with non-isothermal condition can be reduced to a problem with isothermal conditions and thus the solution of the latter will give an approximate solution of the former. The availability of the solution of the corresponding isothermal problem is, therefore, a prerequisite for the application of added thickness method. In the present case the isothermal problem is that of a buried circular cylinder with isothermal wall condition, its axis being parallel to the ground surface. The solution of this problem in Cartesian coordinates as given by Schofield[1] and Eckert[3] is

$$V = Q/4\pi K \cdot log \left[\{ (l + y)^2 + x^2 \} \ / \ \{ (l - y)^2 + x^2 \} \right] \tag{11}$$

where

$$l = \left[(S + m)(S + m + 2r_2) \right]^{1/2} \tag{12}$$

$$Q = 4\pi K \ V_o / log \left[\left\{ 8 (N/D)^2 - 1 \right\} + 4 (N/D) \left\{ 4 (N/D)^2 - 1 \right\}^{1/2} \right] \tag{13}$$

and

S — the depth of the top surface of the pipe from the ground level

r_2 — the radius of the isothermal source

N—the depth of the centre of the source below the ground level, and

D—the diameter of the isothermal source.

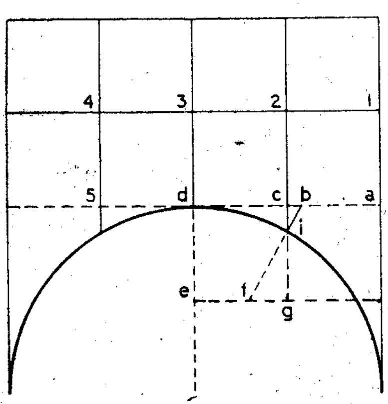

FIG. 2—Nodel points around the circular boundary.

SOLUTION BY THE METHOD OF RELAXATION

A numerical example of the problem has been solved by the relaxation method considering square meshes of length $0 \cdot 25 \ r$ where r is the radius of the buried source. From Fig. 2 we see that there are two types of nodal points which require special treatment in the derivation of relaxation operators; they are points like d lying on the circular boundary and points like c lying very near to it. The residual operators are derived by taking account of the boundary condition. The procedure for obtaining the relaxation operators is illustrated below for a mesh spacing $0 \cdot 5\,r$. The operators for mesh size $0 \cdot 25r$ and $0 \cdot 125r$ were also obtained for the solution but are not presented here for the sake of brevity. The calculations were carried out for mesh sizes of $0 \cdot 25r$ and $0 \cdot 125r$. Since the difference was not significant, values for mesh size of $0 \cdot 25r$ are given here. For the purpose of obtaining the relaxation solution the vertical panes on either side of the pipe parallel to the pipe axis at a distance $4r$ from it were taken to be at zero temperature. It has been verified that moving these boundaries further away would not affect the temperature distribution along the vertical line through the pipe centre.

In Fig. 2, e, f, g are fictitious points, f_b is the normal at i, meeting $c\,a$ at b and eg at f.

Now, if R denotes the residual at any point, then, from the finite difference approximation to the Laplace equation, we get

$$R_d = V_c + V_3 + V_5 + V_e - 4\,V_d \tag{14}$$

where Ve, the fictitious temperature, is eliminated by using the finite difference approximation to the normal derivative $\partial v/\partial_n{}^4$ at d.
Thus

$$-(V_3 - V_e)/2\lambda = h\,(V_o - V_d) \tag{15}$$

where λ is the mesh length and $h = H/K$

$$V_e = V_3 + 2\lambda h\,V_o - 2h\lambda\,V_d \tag{16}$$

Substituting this value in (14), we get

$$R_d = V_c + V_3 + V_5 - (4 + 2\,\lambda\,h)\,V_d + 2\lambda h\,V_o \tag{17}$$

Now

$$R_c = V_a + V_2 + V_d + V_g - 4\,V_c \tag{18}$$

Here V_g, the fictitious temperature is to be eliminated using the condition

$$-(V_b - V_f)/f\,b = h\,(V_o - V_i) \tag{19}$$

The values of V_b and V_f are got by linear interpolation, *i.e.*

$$V_b = V_a + (V_c - V_a)\,b\,a/\lambda \tag{20}$$

$$V_f = V_g + (V_e - V_g)\,fg/\lambda \tag{21}$$

and V_e is known from (16)

On substitution of these in the equation (19) and further simplification, we get

$$A.\,V_g = -fg\,V_3 + c\,b\,V_a + (b\,a - f\,b.\,h.\,g\,i)\,V_c + 2fg.\lambda.h.\,V_d + \lambda\,h\,(f\,b - 2fg)\,V_o \tag{22}$$

where $A \equiv f\,e + f\,b.\,h.\,c\,i$

Substituting this value of V_g in equation (18), we get

$$R_c = (1 + c\,b/A)\,V_a + V_2 + (1 + 2fg.\lambda.h/A)\,V_d - fg\,V_3/A - \{4 - (b\,a - f\,b.\,h.\,g\,i)/A\}\,V_c + \lambda\,h\,(f\,b - 2fg)\,V_o/A \tag{23}$$

TABLE 1

A COMPARISON OF THE VALUES OF TEMPERATURES AT VARIOUS POINTS ON THE BOUNDARY BY THE TWO METHODS

V/V_o	$\theta=0°$	$\theta=15°$	$\theta=30°$	$\theta=45°$
Added thickness method	·814	·816	·818	·823
Relaxation method	·807	·812	·815	·820

The Numerical example :

The values of constants chosen for the example are

$r_1 = 5$ cm, $N=15$ cm, $S=10$ cm

$K=0\cdot0045$ cgs, $H=0\cdot0035$ cgs, and $m=1\cdot2857$ cm

A comparison of the values of temperatures at various points on the boundary of the pipe obtained by the two methods is given in Table 1.

The temperatures at different points along the vertical line to the ground through the pipe centre are compared in Fig. 3.

CONCLUSION

It is clear that the results obtained by the added thickness method are quite close to those obtained by relaxation method. The added thickness method is thus found to give quick and quite accurate result to the steady state heat flow problem with non-isothermal condition at a circular boundary. The method can be applied only to those problems where the added thickenss $K/H<r_1$.

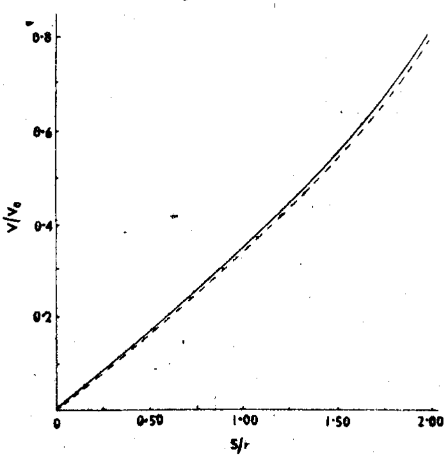

FIG. 3—Comparison of results obtained by added thickness and relaxation methods.

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