EFFECTS OF SMALL VARIATION IN DESIGN PARAMETERS ON THE PERFORMANCE OF A MULTISTAGE ROCKET

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The effects of small perturbations in the design parameters of a multistage rocket on its flight performance have been investigated. To get the best combinations of design parameters, the advantage of a proposed change in some parameter in one of the stages over a change in another parameter in the same or some other stage so as to give a specified all-burnt velocity has also been studied. Three cases have been examined and the changes in all-burnt height have been calculated in each case. The results are of practical importance in the preliminary design of an optimum rocket.

For a rocket already in existence or to be designed, the study of the effects of small perturbations in the various design parameters on the overall performance is of considerable importance. Seifert and Summerfield[1] have mentioned this problem in a qualitative way in the very simplified case of zero drag and gravity. In a recent paper the author[2] discussed the probem analytically for a single stage rocket but the velocity attained with a single stage rocket is severely limited and so is the height or the range attained. In order to overcome this difficulty, use is made of multistage rocket consisting of a number of stages each being a complete rocket in itself. Thus there arises the necessity of finding the effects of small changes or errors in the various design parameters such as payload weight, specific impulses, propellant weights, structure weights and thrusts etc. in any one of the stages on the performance parameters | velocity and all-burnt height. As the development of a new rocket vehicle is a lengthy process, it sometimes happens that multistage vehicles are made up of available rockets (which we may call different stages) which are dissimilar in mass ratios, specific impulses, structure weights and thrusts etc. Also since the range achieved by a missile depends upon the conditions at all-burnt and particularly upon the all-burnt velocity, it is necessary to keep it fixed in order to achieve a given range. Thus it is of interest to study the changes required in some parameter in any one of these rockets when some other parameter in some other rocket is varied so as to get a specified all-burnt velocity. In this paper we will discuss only three cases for fixing the all-burnt velocity, *i.e.* by changing (*i*) propellant weight in one of the stages, (*ii*) thrust in one of the stages and (*iii*) the payload. In every case the height or range attained at all-burnt will be affected; and these changes are found for all-burnt height of a vertically ascending rocket, in all the above three cases when there are separate changes in other design factors. The results obtained are of practical importance for the preliminary design of an optimum rocket.

Effect of small changes in the parameters

The primary factors contributing to the velocity and height at all-burnt are: payload weight, propellant weights, structure weights, specific impulses, thrusts and time of burning for each stage. Assuming thrust to be constant in each stage, gravity to be constant throughout and negligible resistance due to earth's atmosphere, the velocity and height attained uptill all-burnt are given by the following expressions:

$$V_b = \sum_{i=1}^{N} \left[gI_{Sp,i} \log r_i - gt_{b,i} \right] \tag{1}$$

$$h_b = \sum_{i=1}^{N} \left[gI_{Sp,i}\, t_{b,i} \left(1 + \frac{\log r_i}{1-r_i} \right) - \frac{1}{2}\, gt^2_{b,i} \right] \tag{2}$$

where

$$r_i = \frac{M_{o,i} + M_L}{M_{s,i} + M_{o,i+1} + M_L} \tag{3}$$

$M_{o,i}$ = total initial mass of the ith step rocket.

$M_{s,i}$ = structure mass of the ith stage.

M_L = payload mass

$I_{Sp,i}$ = specific impulse of the ith stage.

$t_{b,i}$ = time of burning of the propellant in the ith stage.

N = total number of stages

g = acceleration due to gravity, constant

From (3) we can deduce that

$$\frac{M_{p,i}}{M_{o,i}} = \frac{r_i - 1}{r_i} \tag{4}$$

where $M_{p,i}$ = propellant mass of the ith stage. Also the time of burning for each stage in terms of rocket parameters is given by

$$t_{b,i} = \frac{M_{p,i}}{F_i} gI_{Sp,i} \tag{5}$$

in which F_i is the constant thrust in the ith stage.

With the help of equations (1) to (5), we can find the changes in the velocity and height at all-burnt due to changes in payload weight, structure weights, propellant weights, specific impulses and thrusts in any one of the stages, say nth. The required changes in all-burnt velocity are given by

$$\frac{\partial V_b}{\partial M_L} = \sum_{i=1}^{N} - \frac{gI_{Sp,i}}{M_{o,i}} (r_i - 1) \tag{6}$$

$$\frac{\partial V_b}{\partial M_{s,n}} = \sum_{i=1}^{n} - \frac{gI_{Sp,i}}{M_{o,i}} (r_i - 1) \tag{7}$$

$$\frac{\partial V_b}{\partial M_{p,n}} = \sum_{i=1}^{n-1} - \left[\frac{g I_{Sp,i}}{M_{o,i}} (r_i - 1) \right] + \frac{g I_{Sp,n}}{M_{o,n}} \left(1 - \frac{W_{o(n)}}{F_n} \right) \tag{8}$$

$$\frac{\partial V_b}{\partial I_{Sp,n}} = g \left[\log r_n - \frac{W_{o(n)}}{F_n} \; \frac{r_n - 1}{r_n} \right] \tag{9}$$

$$\frac{\partial V_b}{\partial F_n} = g \; \frac{I_{Sp,n}}{F_n} \cdot \frac{W_{o(n)}}{F_n} \cdot \frac{r_n - 1}{r_n} \tag{10}$$

In the above $W_{o(n)}$ means the total initial weight of the nth step rocket. Here we see that all-burnt velocity decreases (or increases) with an increase (or decrease) in M_L and $M_{s,n}$ while the reverse happens with changes in $I_{sp,n}$ and F_n. From (8) we observe that if $n=1$, then there is an increase in V_b due to an increase in $M_{p,1}$ and the change is given by

$$\frac{\partial V_b}{\partial M_{p,1}} = \frac{g I_{Sp,i}}{M_{o,i}} \left(1 - \frac{W_{o(1)}}{F_i} \right) \tag{11}$$

But if $n \geqslant 2$, then V_b may decrease due to an increase in $M_{p,n}$ provided that

$$\frac{W_{o(n)}}{F_n} > 1 - \frac{\sum_{i=1}^{n-1} \frac{I_{Sp,i}}{M_{o,i}} (r_i - 1)}{I_{Sp,n} / M_{o,n}} \tag{12}$$

Now we consider a specific example of a three stage-rocket for which $r_1 = 3$, $r_2 = 2 \cdot 5$, $r_3 = 2$, initial weights $W_{o(1)} = 150{,}0000$ lb, $W_{o(2)} = 400{,}000$ lb., $W_{o(3)} = 100{,}000$ lb., payload weight=10,000 lb, $W_{o(1)}/F_1 = 0 \cdot 25$, $W_{o(2)}/F_2 = 0 \cdot 4$, $W_{o(3)}/F_3 = 0 \cdot 5$, specific impulses $I_{sp,1} = 300$ sec, $I_{Sp,2} = 400$ sec and $I_{Sp,3} = 350$ sec. For this example we see that condition (12) is satisfied and for an increase in $M_{p,3}$ the velocity at all-burnt does not increase, as it does due to increase in $M_{p,1}$ or $M_{p,2}$ but it decreases. Table 1 gives the percentage changes in V_b and h_b due to one per cent change in the various parameters for the example considered.

TABLE 1

PERCENTAGE CHANGES IN V_b AND h_b DUE TO ONE PERCENT CHANGE IN VARIOUS DESIGN PARAMETERS

Parameter	Stage (n)					
	1		2		3	
	V_b	h_b	V_b	h_b	V_b	h_b
$M_{s,n}$	−0·0567	−0·0303	−0·1617	−0·1600	−0·3063	−0·3276
$M_{p,n}$	0·2127	0·3502	0·3403	0·5729	−0·0106	0·2242
$I_{sp,n}$	0·3965	0·5146	0·3836	0·9652	0·2199	0·5201
F_n	0·0709	−0·1989	0·1361	−0·2674	0·1240	−0·0813

The percentage change in V_b due to one per cent change in M_L for this example is $-0\cdot0766$. If all the parameters in any one of the stages are varied then the total change in all-burnt velocity is the sum total of (6) to (10). If the same parameter for all stages is varied then the total change is given by the repeated application of the expression giving that particular change.

Again the effects on all-burnt height due to small errors or variations in the payload weight, structure weights, propellant weights, specific impulses and thrusts in any one of the stages are respectively given by

$$\frac{\partial h_b}{\partial M_L} = \sum_{i=1}^{N} \left[\frac{g^2 I^2 s_{p,i}}{F_i} \quad \frac{r_i - 1}{r_i} \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right] \tag{14}$$

$$\frac{\partial h_b}{\partial M_{s,n}} = \sum_{i=1}^{n} \left[\frac{g^2 I^2 s_{p,i}}{F_i} \quad \frac{r_i - 1}{r_i} \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right] \tag{15}$$

$$\frac{\partial h_b}{\partial M_{p,n}} = \sum_{i=1}^{n-1} \left[\frac{g^2 I^2 s_{p,i}}{F_i} \quad \frac{r_i - 1}{r_i} \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right] + \frac{g^2 I^2 s_{p,n}}{F_n} \quad \frac{r_n - 1}{r_n} \left(1 - \frac{W_{o(n)}}{F_n} \right) \tag{16}$$

$$\frac{\partial h_b}{\partial I s_{p,n}} = g I s_{p,n} \frac{W_{o(n)}}{F_n} \frac{r_n - 1}{r_n} \left[2 \left(1 + \frac{\log r_n}{1 - r_n} \right) - \frac{W_{o(n)}}{F_n} \quad \frac{r_n - 1}{r_n} \right] \tag{17}$$

$$\frac{\partial h_b}{\partial F_n} = - \frac{g I^2 s_{p,n}}{F_n} \frac{r_n - 1}{r_n} \frac{W_{o(n)}}{F_n} \left[\left(1 + \frac{\log r_n}{1 - r_n} \right) - \frac{W_{o(n)}}{F_n} \quad \frac{r_n - 1}{r_n} \right] \tag{18}$$

From (14), (15) and (17) it is clear that h_b increases (or decreases) due to a decrease (or increase) in M_L, $M_{s,n}$ and F_n while from (17) we see that the reverse happens for a change in $I s_{p,n}$. Also it is obvious that if the rocket is of one stage then the change in h_b due to equal changes in M_L and M_s are equal. Also from (16) we find that if $n=1$, then there is an increase in h_b due to an increase in $M_{p,1}$ and the change is given by

$$\frac{\partial h_b}{\partial M_{p,1}} = \frac{g^2 I^2 s_{p,1}}{F_1} \quad \frac{r_1 - 1}{r_1} \left(1 - \frac{W_{o(1)}}{F_1} \right) \tag{19}$$

and in this case we can make use of Table 2† to find the percentage increase in h_b due to one per cent increase in $M_{p,1}$ for different values of r_1 and $F_1 / W_{o(1)}$. If, however, $n \geqslant 2$, there is a chance of h_b being decreased due to an increase in $M_{p,n}$ provided that

$$\frac{W_{o(n)}}{F_n} > 1 + \frac{\displaystyle\sum_{i=1}^{n-1} \frac{I^2 s_{p,i}}{F_i} \quad \frac{r_i - 1}{r_i} \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\}}{\displaystyle\frac{I^2 s_{p,n}}{F_n} \quad \frac{r_n - 1}{r_n}} \tag{20}$$

For the given example, Table 1 gives the percentage changes in h_b for unit per cent changes in the various parameters.

† This refers to Table 2 reference 2.

Also the percentage change in h_b due to unit per cent change in M_L is—0·0819. For the case when all the parameters in a particular stage are varied, the total change in h_b is given by summing the changes given by (14) to (18). In case the same prameter is varied in all the stages, the changes are given by summing the same expression from 1 to n.

Corrections required in payload weight, propellant weights and thrusts due to changes in different design variables for maintaining a constant all-burnt velocity

Now in order to seek for the best combinations of design parameters, we have to find the advantage of a change in one parameter over a change in another parameter. While designing a multistage rocket we search for these combinations by finding the change required in a particular parameter in a given stage when another given parameter in the same stage or in some other known stage undergoes a small change while maintaining the all-burnt velocity constant. Some possible convenient ways by which all-burnt velocity can be made invariable are to make necessary corrections in (i) propellant weight in the same stage or some other suitable stage, (ii) thrust in the same stage or some other suitable stage and (iii) payload weight, when other parameters in the same or different stage undergo alterations. The corrections required in propellant weight, thrust and payload weight have been found in each case and the overall effect on the all-burnt height has also been calculated.

Case 1—One convenient device by which we can make the all-burnt velocity invariable is to make the necessary changes in the charge weight of a particular stage. The charge weight can be altered in the same stage in which the other variable is in error which is causing a change in the velocity or in any other stage below or above it; thus, if there is a change in $M_{s,n}$ of the nth stage, which we shall call the error stage, we may alter the charge weight of the tth stage which we shall call the balancing stage where t may have any value from 1 to N. The subscript $(\quad)_{v_b,\, m_{p,t}}$ is used to mean that all-burnt velocity is being maintained at a fixed value by making the necessary adjustments in $M_{p,t}$. The necessary changes in $M_{p,t}$ due to changes in M_L, $M_{s,n}$, $I_{sp,n}$ and F_n are respectively as follows :

$$\frac{\partial M_{p,t}}{\partial M_L} = \frac{\sum_{i=1}^{N} \frac{I_{sp,i}}{M_{o,i}} (r_i - 1)}{\sum_{i=1}^{t-1} - \left[\frac{I_{sp,i}}{M_{o,i}} (r_i - 1)\right] + \frac{I_{sp,t}}{M_{o,t}} \left(1 - \frac{W_{o(t)}}{F_t}\right)} \tag{21}$$

$$\frac{\partial M_{p,t}}{\partial M_{s,n}} = \frac{\sum_{i=1}^{n} \frac{I_{sp,i}}{M_{o,i}} (r_i - 1)}{\sum_{i=1}^{t-1} - \left[\frac{I_{sp,i}}{M_{o,i}} (r_i - 1)\right] + \frac{I_{sp,t}}{M_{o,t}} \left(1 - \frac{W_{o(t)}}{F_t}\right)} \tag{22}$$

$$\frac{\partial M_{p,t}}{\partial I_{sp,n}} = - \frac{\log r_n - \dfrac{W_{o(n)}}{F_n} \cdot \dfrac{r_n - 1}{r_n}}{\sum\limits_{i=1}^{t-1} - \left[\dfrac{I_{sp,i}}{M_{o,i}} (r_i - 1) \right] + \dfrac{I_{sp,t}}{M_{o,t}} \left(1 - \dfrac{W_{o(t)}}{F_t} \right)} \quad (23)$$

$$\frac{\partial M_{p,t}}{\partial F_n} = - \frac{\dfrac{I_{sp,n}}{F_n} \cdot \dfrac{W_{o(n)}}{F_n} \cdot \dfrac{r_n - 1}{r_n}}{\sum\limits_{i=1}^{t-1} - \left[\dfrac{I_{sp,i}}{M_{o,i}} (r_i - 1) \right] + \dfrac{I_{sp,t}}{M_{o,t}} \left(1 - \dfrac{W_{o(t)}}{F_t} \right)} \quad (24)$$

It is obvious that whatever the value of n may be, if $t = 1$, we require an increase in $M_{p,1}$ due to an increase in M_L and $M_{s,n}$, but a decrease in $M_{p,1}$ due to an increase in $I_{sp,n}$ and F_n. But if $t \geqslant 2$ the opposite may happen with each provided that

$$\frac{W_{o(t)}}{F_t} > 1 - \frac{\sum\limits_{i=1}^{t-1} \dfrac{I_{sp,i}}{M_{o,i}} (r_i - 1)}{I_{sp,t} / M_{o,t}} \quad (25)$$

which in the particular case when all the specific impulses are equal, reduces to

$$\frac{W_{o(t)}}{F_t} > 1 - M_{o,t} \left\{ \sum_{i=1}^{t-1} \frac{r_i - 1}{M_{o,i}} \right\} \quad (26)$$

and thus depends only on the initial masses of the individual stages and the thrust of the tth stage. Also the effects on height at all-burnt with variations in M_L, $M_{s,n}$, $I_{sp,n}$ and F_n are given by

$$\left(\frac{\partial h_b}{\partial M_L} \right)_{V_b, M_{p,t}} = \sum_{i=1}^{N} \left[\frac{g^2 I^2_{sp,i}}{F_i} \; \frac{r_i - 1}{r_i} \; \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right]$$

$$+ \left[\sum_{i=1}^{t-1} \left\{ \frac{g^2 I^2_{sp,i}}{F_i} \; \frac{r_i - 1}{r_i} \left(1 + \frac{r_i \log r_i}{1 - r_i} \right) \right\} + \frac{g^2 I^2_{sp,t}}{F_t} \; \frac{r_t - 1}{r_t} \left(1 - \frac{W_{o(t)}}{F_t} \right) \right]$$

$$\frac{\sum\limits_{i=1}^{N} \dfrac{I_{sp,i}}{M_{o,i}} (r_i - 1)}{\sum\limits_{=1}^{t-1} - \left\{ \dfrac{I_{sp,i}}{M_{o,i}} (r_i - 1) \right\} + \dfrac{I_{sp,t}}{M_{o,t}} \left(1 - \dfrac{W_{o(t)}}{F_t} \right)} \quad (27)$$

$$\left(\frac{\partial h_b}{\partial M_{s,n}} \right)_{V_b, M_{p,t}} = \sum_{i=1}^{n} \left[\frac{g^2 I^2_{sp,i}}{F_i} \; \frac{r_i - 1}{r_i} \; \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right]$$

$$+\left[\sum_{i=1}^{t-1}\left\{\frac{g^2 I^2 s_{p,i}}{F_i}\frac{r_i-1}{r_i}\left(1+\frac{r_i \log r_i}{1-r_i}\right)\right\}+\frac{g^2 I^2 s_{p,t}}{F_t}\frac{r_t-1}{r_t}\left(1-\frac{W_{o(t)}}{F_t}\right)\right]$$

$$\frac{\sum_{i=1}^{n}\frac{I s_{p,i}}{M_{o,i}}(r_i-1)}{\sum_{i=1}^{t-1}-\left\{\frac{I s_{p,i}}{M_{o,i}}(r_i-1)\right\}+\frac{I s_{p,t}}{M_{o,t}}\left(1-\frac{W_{o(t)}}{F_t}\right)} \tag{28}$$

$$\left(\frac{\partial h_b}{\partial I s_{p,n}}\right)_{V_b,\,M_{p,t}}=g\,I s_{p,n}\,\frac{W_{o(n)}}{F_n}\,\frac{r_n-1}{r_n}\left[2\left(1+\frac{\log r_n}{1-r_n}\right)-\frac{W_{o(n)}}{F_n}\,\frac{r_n-1}{r_n}\right]$$

$$-\left\{\sum_{i=1}^{t-1}\left[\frac{g^2 I^2 s_{p,i}}{F_i}\cdot\frac{r_i-1}{r_i}\times\left(1+\frac{r_i \log r_i}{1-r_i}\right)\right]+\frac{g^2 I^2 s_{p,t}}{F_t}\frac{r_t-1}{r_t}\left(1-\frac{W_{o(t)}}{F_t}\right)\right\}$$

$$\frac{\log r_n-\frac{W_{o(n)}}{F_n}\cdot\frac{r_n-1}{r_n}}{\sum_{i=1}^{t-1}-\left\{\frac{I s_{p,i}}{M_{o,i}}(r_i-1)\right\}+\frac{I s_{p,t}}{M_{o,t}}\left(1-\frac{W_{o(t)}}{F_t}\right)} \tag{29}$$

$$\left(\frac{\partial h_b}{\partial F_n}\right)_{V_b,\,M_{p,t}}=-\frac{g I^2 s_{p,n}}{F_n}\cdot\frac{W_{o(n)}}{F_n}\cdot\frac{r_n-1}{r_n}\left[\left(1+\frac{\log r_n}{1-r_n}\right)-\frac{W_{o(n)}}{F_n}\,\frac{r_n-1}{r_n}\right]$$

$$-\left[\sum_{i=1}^{t-1}\left\{\frac{g^2 I^2 s_{p,i}}{F_i}\frac{r_i-1}{r_i}\times\left(1+\frac{r_i \log r_i}{1-r_i}\right)\right\}+\frac{g^2 I^2 s_{p,t}}{F_t}\frac{r_t-1}{r_t}\left(1-\frac{W_{o(t)}}{F_t}\right)\right]$$

$$\frac{\frac{I s_{p,n}}{F_n}\cdot\frac{W_{o(n)}}{F_n}\cdot\frac{r_n-1}{r_n}}{\sum_{i=1}^{t-1}-\left\{\frac{I s_{p,i}}{M_{o,i}}(r_i-1)\right\}+\frac{I s_{p,t}}{M_{o,t}}\left(1-\frac{W_{o(t)}}{F_t}\right)} \tag{30}$$

Now from (29) we observe that in case the balancing stage is the first stage and since $2\left(1+\frac{\log r_n}{1-r_n}\right)<\log r_n$, and if $\frac{W_{o(n)}}{F_n}\, I s_{p,n}\,\frac{r_n-1}{r_n}<\frac{W_{o(1)}}{F_1}\, I s_{p,1}\,\frac{r_1-1}{r_1}$, then there is bound to be a decrease in h_b for an increase in $I s_{p,n}$. For the example considered this condition is not satisfied. Table 2 gives percentage changes required in $M_{p,t}$ due to one per cent changes in M_L ; $M_{s,n}$; $I s_{p,n}$ and F_n for various values of t for the example considered. Also Table 3 gives the corresponding percentage changes in h_b.

TABLE 2

PERCENTAGE CHANGES IN CHARGE WEIGHTS REQUIRED TO KEEP ALL-BURNT VELOCITY CONSTANT

Error stage parameter	Balancing Stage (i)		
	1	2	3
M_L	0·2080	0·2108	−1·5326
$M_{s,1}$	0·0631	0·0652	−1·2262
M_{s2}	0·1062	0·1122	−3·5648
$M_{s,3}$	0·1767	0·1880	−6·7857
$I_{sp,1}$	−0·1382	−0·1528	8·8740
$I_{sp,2}$	0·3336	0·3195	9·0535
$I_{sp,3}$	0·1580	0·1499	5·1576
F_1	−0·3157	−0·3183	1·2960
F_2	−0·4916	−0·4966	2·6028
F_3	−0·2856	−0·2902	2·5349

TABLE 3

PERCENTAGE CHANGES IN h_b WHEN ALL-BURNT VELOCITY IS MAINTAINED CONSTANT BY MAKING ADJUSTMENTS IN CHARGE WEIGHTS

Error stage parameter	Balancing stage (i)		
	1	2	3
M_L	0·3600	0·2250	−7·2000
$M_{s,1}$	0·2667	0·1667	−5·2000
$M_{s,2}$	0·7600	0·4750	−15·2002
$M_{s,3}$	1·4400	0·8999	−28·8000
$I_{sp,1}$	−1·8639	−1·1649	37·2786
$I_{sp,2}$	−1·8034	−1·1271	36·0697
$I_{sp,3}$	−1·0340	−0·6463	20·6807
F_1	−0·3334	−0·2083	6·6668
F_2	−0·6400	−0·4000	12·8002
F_3	−0·5833	−0·3646	11·6668

Case 2—As mentioned earlier another possible way of keeping the all-burnt velocity constant is by controlling thrust in any one of the stages. Similar expressions as in the previous case can be easily obtained in this case also giving the changes required in F_t and also the changes in h_b due to small errors in M_L, $M_{s,n}$, $M_{p,n}$ and $I_{Sp,n}$. Table 4 gives percentage changes in F_t due to one per cent changes in M_L, $M_{s,n}$, $M_{p,n}$ and $I_{Sp,n}$ and Table 5 gives the corresponding figures for h_b for various values of t in the case of the particular rocket considered.

TABLE 4

PERCENTAGE CHANGES IN THRUST REQUIRED TO KEEP ALL-BURNT VELOCITY CONSTANT

Error Stage Parameter	Balancing Stage (t)		
	1	2	3
M_L	1·0800	0·5625	0·6171
$M_{s,1}$	0·8000	0·4167	0·4571
$M_{s,2}$	2·2800	1·1875	1·3029
$M_{s,3}$	4·3200	2·2500	2·4686
$M_{p,1}$	—3·0000	—1·5625	—1·7143
$M_{p,2}$	—4·8000	—1·5000	—2·7429
$M_{p,3}$	—0·1500	0·0781	0·8570
$I_{Sp,1}$	—5·5910	—2·9123	—3·1953
$I_{Sp,2}$	—5·4097	—2·8179	—3·0916
$I_{Sp,3}$	—3·1020	—1·6157	—1·7726

TABLE 5

PERCENTAGE CHANGES IN h_b WHEN ALL-BURNT VELOCITY IS MAINTAINED CONSTANT BY CONTROLLING THE THRUST

Error Stage Parameter	Balancing Stage (t)		
	1	2	3
M_L	—0·2968	—0·2323	—0·1321
$M_{s,1}$	—0·1894	—0·1417	—0·0674
$M_{s,2}$	—0·6136	—0·4775	—0·2659
$M_{s,3}$	—1·1871	—0·9293	—0·5283
$M_{p,1}$	0·9471	0·7681	0·4896
$M_{p,2}$	1·0703	1·2415	0·7959
$M_{p,3}$	0·1944	0·2033	0·2173
$I_{Sp,1}$	1·6270	1·2935	0·7744
$I_{Sp,2}$	2·0415	1·7188	1·2166
$I_{Sp,3}$	1·1372	0·9522	0·6642

TABLE 6

PERCENTAGE CHANGES IN PAYLOAD FOR KEEPING ALL-BURNT VELOCITY FIXED

Parameter	Stage (n)		
	1	2	3
$M_{s,n}$	—0·7407	—2·1112	—4·0000
$M_{p,n}$	2·7779	4·4450	—0·1390
$I_{Sp,n}$	5·1780	5·0098	2·8724
F_n	0·9260	1·8000	1·6204

TABLE 7

PERCENTAGE CHANGES IN h_b WHEN ALL-BURNT VELOCITY IS KEPT FIXED BY ADJUSTING PAYLOAD WEIGHT

Parameter	Stage (n)		
	1	2	3
$M_{s,n}$	0·0304	0·0130	0·000
$M_{p,n}$	0·1227	0·2090	0·2356
$I_{Sp,n}$	0·0906	0·5549	0·2849
F_n	—0·2748	—0·4130	—0·2140

Case 3—Finally, in case the all-burnt velocity is to be maintained at a given value by making necessary adjustments in the payload weight then, as in earlier cases, we can calculate these required changes in M_L due to changes in other design variables. Table 6 gives the percentage changes in M_L and Table 7 the percentage changes in h_b for the example considered.

Changes in all-burnt height due to small change in all-burnt velocity

Now we discuss the case where V_b is in error or undergoes a small change and these changes in V_b are brought about by small changes or errors in payload weight, structure weights, propellant weights, specific impulses and thrusts in any one of the stages comprising the multistage rocket. Here we will use the subscripts $(\ )_{M_L}$, $(\ )_{M_{s,n}}$, $(\ )_{M_{p,n}}$, $(\ )_{I_{Sp,n}}$ and $(\ )_{F_n}$ to denote that the small error in V_b is caused by errors in M_L, $M_{s,n}$, $M_{p,n}$, $I_{Sp,n}$ and F_n respectively. The following expressions give various changes in all-burnt altitude.

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{M_L} = -\sum_{i=1}^{N}\left[\frac{I^2_{Sp,i}}{F_i}\ \frac{r_i-1}{r_i}\left\{1+\frac{r_i\log r_i}{1-r_i}\right\}\right]\left[\sum_{i=1}^{N}\frac{I_{Sp,i}}{W_{o(i)}}(r_i-1)\right]^{-1} \tag{31}$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{M_{s,n}} = - \sum_{i=1}^{n} \left[\frac{I^2_{Sp,i}}{F_i} \frac{r_i - 1}{r_i} \left\{ 1 + \frac{r_i \log r_i}{1 - r_i} \right\} \right] \left[\sum_{i=1}^{N} \frac{I_{Sp,i}}{W_{o(i)}} (r_i - 1) \right]^{-1} \tag{32}$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{M_{p,n}} = \left[\sum_{i=1}^{n-1} \left\{ \frac{I^2_{Sp,i}}{F_i} \frac{r_i - 1}{r_i} \left(1 + \frac{r_i \log r_i}{1 - r_i} \right) \right\} + \frac{I^2_{Sp,n}}{F_n} \frac{r_n - 1}{r_n} \left(1 - \frac{W_{o(n)}}{F_n} \right) \right] \left[\sum_{i=1}^{n-1} - \left\{ \frac{I_{Sp,i}}{W_{o(i)}} (r_i - 1) \right\} + \frac{I_{Sp,n}}{W_{o(n)}} \left(1 - \frac{W_{o(n)}}{F_n} \right) \right]^{-1} \tag{33}$$

$$\left(\frac{h_b}{\partial V_b}\right)_{I_{Sp,n}} = \frac{W_{o(n)}}{F_n} I_{Sp,n} \frac{r_n - 1}{r_n} \left[2 \left(1 + \frac{\log r_n}{1 - r_n} \right) - \frac{W_{o(n)}}{F_n} \frac{r_n - 1}{r_n} \right] \left[\log r_n - \frac{W_{o(n)}}{F_n} \frac{r_n - 1}{r_n} \right] \tag{34}$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{F_n} = - I_{Sp,n} \left[\left(1 + \frac{\log r_n}{1 \perp r_n} \right) - \frac{W_{o(n)}}{F_n} \frac{r_n - 1}{r_n} \right] \tag{35}$$

Table 8 gives the percentage changes in h_b in this case for the particular mutlistage rocket considered.

Also the change in h_b due to a small change in V_b brought about by a variation in payload mass M_L is 1·0696.

TABLE 8

PERCENTAGE CHANGES IN h_b DUE TO UNIT PER CENT CHANGES IN V_b BROUGHT ABOUT BY ERRORS IN VARIOUS DESIGN PARAMETERS

Parameter	Stage (n)		
	1	2	3
$M_{s,n}$	0·5334	0·9894	1·0696
$M_{p,n}$	1·6466	1·6835	—21·0750
$I_{Sp,n}$	1·2981	2·5162	2·3648
F_n	—2·8060	—1·9645	—0·6552

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