ON THE STABILITY OF A SPINNING PROJECTILE

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Stability problem both for small and large yawing motion of a spinning projectile has been discussed. In the latter case criterion for stability of steady conically yawing motion has been obtained. Particularly it has been proved that with a tilting moment coefficient of the type $\mu(\delta)=\beta\Omega^2/45\,[1-4qs\,(1-\cos\delta)]$ the motion of a shell in steady state is stable like an equivalent top only when $q\geqslant 0$.

A few topics on the stability of a spinning shell have been discussed, particularly, the McShane-Murphy[1] stability conditions and the stability of the steady precessional motion.

In the first part of the paper we have derived McShane-Murphy[1] stability condition for slowly yawing motion from Fowler's[2] dynamical equations, generalised by Rath[3] taking into account the complete aerodynamic force system of Nielsen and Synge[4].

The second part of the paper is devoted to the study of steady conically yawing motion of the shell. In case of broad side fire from an aeroplane the initial oscillations of the shell are such as can not be regarded small as in McShane-Murphy[1] linear theory. For this purpose the coefficient of the aerodynamic-moment (tending to increase the yaw) is treated as an even function of yaw as suggested by Fowler and Lock[5] and the stability problem in the non-linear case is treated in analogy with the common top. This has led us to examine the stability of motion in steady state of the shell. Conditions under which a shell behaves like a common top for stability purposes have been obtained and it has been shown that the steady precessional motion of the shell is not essentially stable as that of a common top. Finally the stability has been considered of the normal motion—for which the axis of the shell coincides with the direction of motion of its centre of mass. The stability condition analogous to a sleeping top holds good even in this non-linear case as already proved by Fowler and Lock[2].

SYMBOLS AND NOTATIONS

$\vec{Z}$ = unit vector in the direction of the axis of the projectile.

$\vec{X}$ = unit vector along the tangent to the path of the c.m. of the projectile.

δ = angle of yaw.

ρ = air density.

v = speed of the projectile c.m.

$\vec{\omega}$ = angular velocity vector of the projectile axis.

d = calibre of the projectile.

N = angular speed of the projectile round its axis.

A, B = axial and transverse moment of inertia of the projectile.

$\Omega = AN/B$

a = local velocity of sound

$\vec{\xi}$ = vector yaw of the projectile.

θ = inclination of the velocity vector of the projectile with the plane of horizon.

$\theta_\circ$ = value of θ in the corresponding normal position of the projectile.

ψ = inclination of the velocity vector of the projectile c.m with the plane of fire, commonly known as the angle of drift.

m = mass of the projectile.

$\vec{g}$ = acceleration vector due to gravity

PART I

McShane-Murphy stability condition

The aerodynamic forces and moments acting on the projectile are given by[3]

$$\vec{R} = -\rho\, d^2\, v^2 f_R\, \vec{X} \quad = -m\, f_1\, \vec{X} \tag{1}$$

$$\vec{L} = \rho\, d^2\, v^2\, f_L\, \vec{X} \times \left(\vec{Z} \times \vec{X}\right) = m\, v\, f_2 \left\{\vec{Z} - \vec{X}\,(\vec{Z}\cdot\vec{X})\right\}^{\dagger} \tag{2}$$

$$\vec{M} = \rho\, d^3\, v^2 f_M\, \vec{X} \times \vec{Z} = f_4\, \vec{X} \times \vec{Z} \tag{3}$$

$$\vec{I} = \rho\, d^4\, v\, N\, f_I\, \vec{Z} \quad = -A\, N\, f_6\, \vec{Z} \tag{4}$$

$$\vec{J} = \rho\, d^4\, v\, N\, f_J\, \vec{Z} \times (\vec{Z} \times \vec{X}) = A\, N\, f_5 \left\{\vec{Z}\,(\vec{Z}\cdot\vec{X}) - \vec{X}\right\} \tag{5}$$

$$\vec{K} = \rho\, d^3\, v\, N\, f_K\, \vec{Z} \times \vec{X} = m\, v\, f_3\, \vec{Z} \times \vec{X} \tag{6}$$

$$\vec{H} = \rho\, d^4\, v\, \vec{\omega}\, f_H \quad = -B\, f_7\, \vec{\omega} \tag{7}$$

$$\vec{P} = m\, v\, f_8\, \vec{Z} \times \vec{Z}' \tag{8}$$

$$\vec{Q} = -m\, v\, f_9\, \vec{Z}' \tag{9}$$

$$\vec{T} = -B\, f_{10}\, \vec{Z}', \qquad ('\text{—}' = d/dt) \tag{10}$$

In these equations f_i's are functions of δ, v/a, $\omega\, d/v$, $N\, d/v$ etc; the function f_3 contains also a factor N. For practical purposes, however, we treat them as functions of δ and v/a only.

† A cross ($\times$) and a dot ($\cdot$) in the above relations signify as usual vector and scalar products of vector quantities.

The general vector equation of motion

Consider an orthogonal right handed system of moving axis, whose origin is the centre of mass of the shell and which moves in any manner with an angular velocity $\vec{\theta}$ relative to some inertial frame. Relative to the moving frame of reference, one can write the vector equation of motion of the shell as follows:

$$\Omega' = -f_6\, \Omega \tag{11}$$

$$\begin{aligned} \Omega \vec{Z'} + \vec{Z} \times \vec{Z''} - (2\vec{Z'} - \vec{Z} \times \vec{\theta} + f_7 \vec{Z})\ (Z \cdot \theta) \\ - \vec{Z}\ (\vec{Z} \cdot \vec{\theta'}) + \vec{\theta'} - \Omega\ (\vec{Z} \times \vec{\theta}) \\ = f_4 \vec{X} \times \vec{Z} + \Omega\, f_5 \left\{ \vec{Z}\ (\vec{Z} \,.\, \vec{X}) - \vec{X} \right\} \\ - f_7\ (\vec{Z} \times \vec{Z'} + \vec{\theta}) - f_{10}\ (\vec{Z'} - \vec{Z} \times \vec{\theta}) \end{aligned} \tag{12}$$

for the angular motion of the shell and

$$v' = \vec{g} \cdot \vec{X} - f_1 + f_8\, S_1 - f_9\, S_2 \tag{13}$$

where $S_1 = \vec{Z} \times \vec{Z'} \cdot \vec{X} - (\vec{Z} \cdot \vec{\theta})\, \vec{Z} \cdot \vec{X} + \vec{\theta} \cdot \vec{X}$ (14)

$$S_2 = \vec{Z'} \cdot \vec{X} - \vec{Z} \times \vec{\theta} \cdot \vec{X} \tag{15}$$

$$\begin{aligned} \vec{X'} - \vec{X} \times \theta = \left\{ \vec{g} - \vec{X}\ (\vec{g} \cdot \vec{X}) \right\} \Big/ v + f_2 \left\{ \vec{Z} - \vec{X}\ (\vec{Z} \cdot \vec{X}) \right\} \\ + f_3\, \vec{Z} \times \vec{X} - \Big[(f_8\, S_1 - f_9\, S_2)\ \vec{X} \\ + f_9\ (\vec{Z'} - \vec{Z} \times \vec{\theta}) - f_8\ (\vec{Z} \times \vec{Z'} - (\vec{Z} \cdot \vec{\theta})\ \vec{Z} - \vec{\theta}) \Big] \end{aligned} \tag{16}$$

for the rectilinear motion of the shell.

Choice of a moving coordinate system

Choose O-123 a set of right handed orthogonal cartesian co-ordinate axes, such that $O3$ coincides with $\vec{X}$ and $O1$ is in a vertical plane containing $O3$, and $O2$ is at right angle to $O1$.

The stationary co-ordinate system O-xyz at the gun position is such that Ox is along the direction of fire Oy vertically upwards and Oz to the right of the gunner.

Let (Z_1, Z_2, Z_3) be the components of $\vec{Z}$ along the moving axis O-123 and θ, ψ the inclinations of the vector $\vec{X}$ with the fixed plane xz and xy so that we have

$$\vec{Z} = (Z_1,\ Z_2,\ Z_3)$$

$$\vec{X} = (0,\ 0,\ 1)$$

$$\vec{\theta} = (-\psi',\ \theta',\ 0)$$

The above components of the vector relate to moving axis O-123.

The Kelley-McShane equation of yawing motion

The complete dynamical equations of a spinning shell in scalar form are now

$$v' = -g \sin\theta - f_1 + f_8 S_1 - f_9 S_2 \tag{17}$$

$$\theta' = -g\cos\theta / v + f_2 Z_1 + f_3 Z_2 - \Big[f_9 (Z'_1 + Z_3 \theta') - f_8 \big\{ (Z_2 Z_3' - Z_3 Z_2') + Z_1 (Z_1 \psi' - Z_2 \theta') + \psi' \big\} \Big] \tag{18}$$

$$\psi' = -f_3 Z_1 + f_2 Z_2 - \Big[f_9 (Z_2' + Z_3 \psi') - f_8 \big\{ (Z_3 Z_1' - Z_1 Z_3') + Z_2 (Z_1 \psi' - Z_2 \theta') - \theta' \big\} \Big] \tag{19}$$

$$\Omega' = - f_6 \Omega \tag{20}$$

$$\begin{aligned} & Z_2 Z_3'' - Z_2'' Z_3 + \Omega Z_1' - (2 Z_1' + Z_3 \theta' + f_7 Z_1)(Z_2 \theta' - Z_1 \psi') \\ & - Z_1 (Z_2 \theta'' - Z_1 \psi'') - \psi'' + \Omega Z_3 \theta' \\ & = -f_4 Z_2 + \Omega f_5 Z_1 Z_3 - f_7 (Z_2 Z_3' - Z_2' Z_3 - \psi') - f_{10} (Z_1' + Z_3 \theta') \end{aligned} \tag{21}$$

$$\begin{aligned} & Z_3 Z_1'' - Z_3'' Z_1 + \Omega Z_2' - (2 Z_2' + Z_3 \psi' + f_7 Z_2)(Z_2 \theta' - Z_1 \psi') \\ & - Z_2 (Z_2 \theta'' - Z_1 \psi'') + \theta'' + \Omega Z_3 \psi' \\ & = f_4 Z_1 + \Omega f_5 Z_2 Z_3 - f_7 (Z_3 Z_1' - Z_3' Z_1 + \theta') - f_{10} (Z_2' + Z_3 \psi') \end{aligned} \tag{22}$$

and

$$Z_3 = (1 - Z_1^2 - Z_2^2)^{\frac{1}{2}}$$

For slowly yawing motion $Z_1 \psi' - Z_2 \theta'$, $Z_2 \theta' - Z_1 \psi'$ and $Z_2 \theta'' - Z_1 \psi''$ are neglected and $Z_3 = 1$ so that the preceeding equations of motion, after linearization and approximating by means of normal equations† are

$$v' = -g \sin\theta - f_1$$

$$\theta' = -g \cos\theta / v + f_2 Z_1 + f_3 Z_2 - f_9 (Z_1' + \theta') - f_8 (Z_2' - \psi')$$

$$\psi' = -f_3 Z_1 + f_2 Z_2 - f_9 (Z_2' + \psi') + f_8 (Z_1' - \theta')$$

$$\Omega' = - f_6 \Omega$$

$$\left.\begin{aligned} -Z_2'' + \Omega Z_1' - \psi'' + \Omega \theta' &= -f_4 Z_2 + \Omega f_5 Z_1 + f_7 (Z_2' + \psi') \\ &\quad - f_{10} (Z_1' + \theta') \\ Z_1'' + \Omega Z_2' + \theta'' + \Omega \psi' &= f_4 Z_1 + \Omega f_5 Z_2 - f_7 (Z_1' + \theta') \\ &\quad - f_{10} (Z_2' + \psi') \end{aligned}\right\} \tag{23}$$

In the above equations putting

$$\xi = Z_1 + i Z_2 \qquad i = (-1)^{\frac{1}{2}}$$

† normal equations are $\theta_o = -g \cos\theta / V$, $v_o' = -g \sin\theta - f_1$

$$\chi = \theta + i\,\psi$$
$$f = f_2 - i\,f_3$$
$$l = f_9 - i\,f_8$$

we have

$$\xi'' - i\,\Omega\,\xi' + \chi'' - i\,\Omega\,\chi' = f_4\,\xi - i\,\Omega\,f_5\,\xi - f_7\,\xi' - f_7\,\chi' + i\,f_{10}\,\xi' + i\,f_{10}\,\chi' \tag{24}$$

and

$$\chi' = -\,g\,\cos\theta/v + f\,\xi - l\,\xi' \tag{25}$$

$$\chi'' = \theta_\circ'' + f\,\xi' - l\,\xi'' \qquad (\theta_\circ' = -\,g\,\cos\theta/v) \tag{26}$$

after neglecting quantities of small order such as $f'\,Z_K$ $(k = 1, 2)$ and $l'\,Z'_K$ $(k = 1, 2)$ for small yawing motion.

Now making use of (25) and (26) in (24), after neglecting products such as $f_i\,f_j$ for small yawing motion, we have

$$\xi'' + \xi'(-\,i\,\Omega + f_7 - i\,f_{10} + f) + \xi\,(-\,f_4 + i\,\Omega\,f_5 - i\,\Omega\,f) + g\,\cos\theta/v\,\{\,-\,2g\,\sin\theta/v - f_1/v - (f_7 - i\,f_{10} - i\,\Omega)\,\} = 0 \tag{27}$$

In the above equations by substituting

$$k^2 = B/md^2\,, \qquad \nu = Nd/v\,, \quad \epsilon = \rho d^3/m$$

and hence indentifying with McShane[6]

$$f_1 = (v^2/d)\;\phi_R \qquad\qquad f_5 = (v/d)\;(B/A)\;\phi_J\;K^{-2}$$
$$f_2 = (v/d)\;\phi_L \qquad\qquad f_6 = (v/d)\;(B/A)\;\phi_I\;K^{-2}$$
$$f_3 = (v/d)\;\nu\;\phi_K \qquad\qquad f_7 = (v/d)\;\phi_{1H}\;K^{-2}$$
$$f_4 = (v/d)^2\;\phi_m\;K^{-2}, \qquad \text{and } f_{10} = (v/d)\;\nu\;\phi_{XT}$$

and making a change of the independent variable through

$$p = \int (v/d)\,dt$$

where p is the dimensionless arc length of the trajectory of the shell, we have

$$\ddot{\xi} + [\,\phi_H\,k^{-2} + \phi_L - \phi_R - g\,d\,\sin\theta/v^2 - i\,\nu\,(A/B + \phi_K + k^{-2}\phi_{XT})\,]\,\dot{\xi} + [\,-\,\phi_m\,k^{-2} - (A/B)\;\phi_K\,\nu^2 - i\,\nu\,\{\,(A/B)\;\phi_L - k^{-2}\;\phi_J\,\}\,]\,\xi + g\,d\,v^{-2}\cos\theta\,\{\,-\,2\,g\,d\,v^{-2}\sin\theta - \phi_R - k^{-2}\;\phi_H + i\,\nu\,k^{-2}\phi_{XT} + i.\,(A/B).\,\nu\} \tag{28}$$

which is the Kelley McShane[6] equation of slowly yawing motion of a spinning shell. Here the overhead dot implies derivative w.r.t. p.

Since in the treatment of Kelley and McShane[6] it has been shown that there is no contribution of the term containing ϕ_{XT} to stability and also one knows that for ordinary projectiles with small-yaw motion $k^{-2}\;\phi_{XT} << A/B$, we write the homogeneous part of the equation as

$$\ddot{\xi} + (\,H + J_g - i\,\bar{\nu}\,)\,\dot{\xi} + (\,-\,M - i\,\bar{\nu}\;T\,)\;\xi = 0 \tag{29}$$

where

$$\left.\begin{aligned} H &= \phi_L - \phi_R + k^{-2}\,\phi_H \\ J_g &= -\ g d \bar{\nu}^{-2} \sin\theta \\ \bar{\nu} &= (A/B)\,\nu \\ M &= k^{-2}\,\phi_m + (A/B)\,\phi_K \\ T &= \phi_L - (B/A)\,k^{-2}\,\phi_J \end{aligned}\right\} \tag{30}$$

For discussion of stability we need to examine the transient solutions of (28) *i.e.*, the solution of (29). For this we need the equation

$$\dot{\bar{\nu}} = (D - J_g)\,\bar{\nu} \tag{31*}$$

where

$$D = \phi_R - \phi_I\ md^2/A. \tag{32}$$

Approximate solution of the equation of yaw

With the transformation

$$\xi = q\ exp\ 1/2 \int_0^p -(H + J_g - i\,\bar{\nu})\,dp \tag{33}$$

(29) reduces to
$$\ddot{q} - r^2\,q = 0 \tag{34}$$

and with
$$W = (log\ \dot{q}) \tag{35}$$

the equation (29) reduces to

$$\dot{W} + W^2 - r^2 = 0 \tag{36}$$

where

$$4\,r^2 = [\,4M - \bar{\nu}^2 + (H + J_g)^2 + 2\,i\,\bar{\nu}\,(2\,T - H - D)\,]$$

Noting that the aerodynamic coefficients ϕ_i's and $\bar{\nu}$ are slowly varying functions of p and therefore r approximately constant, so that approximate solution of (36) may be taken as
$$W = \pm\, r - \dot{r}/r \tag{37}$$

using a W.B.J.K. approximation[7]. Hence the transient solutions are given by

$$\xi = K_1\,exp\ 1/2 \int_0^p \left[-H - J_g + i\,\bar{\nu} - \epsilon + \left\{ -\,\bar{m} + 2\,i\,\bar{\nu}\,(\,2\,T - H - D\,) \right\}^{\frac{1}{2}} \right] dp$$

$$+ K_2\,exp\ 1/2 \int_0^p \left[-H - J_g + i\,\bar{\nu} - \epsilon - \left\{ -\bar{m} + 2\,i\,\bar{\nu}\,(2\,T - H - D) \right\}^{\frac{1}{2}} \right] dp \tag{38}$$

where K_1, K_2 are constants of integration

*(31) obtained from (20) through $\Omega' = -f_6\,\Omega$

i.e., $(AN/B)' = -(v/d)\,(B/A)\,\phi_I\,\bar{k}^{2}\,(A/B)\,N$ and $\dfrac{1}{\bar{\nu}} = v\,[\,\phi_R - (md^2/A)\,\phi_I + g d\,\bar{v}^{-2}\,\sin\theta\,]$

and

$$\epsilon = \dot{r}/r = (D - J_g)\ \bar{\nu}\left[\left\{\ \bar{\nu} - i\,(2\,T - H - D)\ \right\}\Big/\left\{\ \bar{m} - 2\,i\,\bar{\nu}\,(2\,T - H - D)\right\}\right]$$

$$= (\ D - J_g\ )\ (\ \epsilon_1 + i\ \epsilon_2) \qquad \text{say}$$

$$\doteq (\ D - J_g\ )\ \epsilon_1 \qquad (39)$$

Since $(\ D - J_g\ )\ \epsilon_1$ is small compared to $\bar{\nu}$ it may be neglected.

We note

$$\epsilon_1 = \bar{\nu}^2\left[\ \bar{m} + 2\ (\ 2\ T - H - D\ )^2\right]\Big/\left\{\ \bar{m}^2 + 4\ \bar{\nu}^2 (2\ T - H - D)^2\right\} \qquad (40)$$

and

$$\bar{m} = \bar{\nu}^2 - 4\ (\ md^2/B\ )\ \phi_M - (\ H - J_g\ )^2 \qquad (41)$$

when $\quad |\ \bar{m}\ | >>\ |\ 2\ T - H - D\ |$ and since numerically

$$\bar{\nu} < 1/20 \qquad \text{we have} \qquad \epsilon_1 = \bar{\nu}^2\ /\ \bar{m} \qquad (42)$$

McShane-Murphy stability conditions

According to Murphy a projectile is dynamically stable if the yaw described by (38) does not increase with time. This happens if the following conditions are satisfied.

$$H + \bar{\epsilon} \geqslant \left|\ R_e\ \left\{\left[\ -\ \bar{m} + 2\ i\ \bar{\nu}\ (\ 2\ T - H - D\ )\right]^{\frac{1}{2}}\right\}\ \right| \qquad (43)$$

Here R_e stands for the real part of the expression inside the curly bracket and

$$\bar{\epsilon} = J_g + (\ D - J_g\ )\ \epsilon_1 \qquad (44)$$

putting $\qquad a = -\ \bar{m} \quad$ and $\ b = 2\ \bar{\nu}\ (\ 2\ T - H - D\ )$

and using De-Moivers theorem the conditions (43) imply

$$H + \bar{\epsilon} \geqslant +\left[\ \left\{\ (\ a^2 + b^2\ )^{\frac{1}{2}} + a\right\}\Big/\ 2\ \right]^{\frac{1}{2}} \qquad (45)$$

i.e.

$$2\ (\ H + \bar{\epsilon}^{\,2}\ ) \geqslant (\ a^2 + b^2\ )^{\frac{1}{2}} + a$$

and

$$H + \bar{\epsilon} \geqslant 0$$

These in turn give rise to two sets of conditions. Writing them separately we have

$$\left.\begin{array}{c} H + \bar{\epsilon} > 0 \\ \left\{\ 2\ \ (\ H + \bar{\epsilon})^2 - a\right\}^2 \geqslant a^2 + b^2 \end{array}\right\} \qquad (46)$$

and

$$\left.\begin{array}{r} H + \bar{\epsilon} = 0 \\ b = 0 \\ a \leqslant 0 \end{array}\right\} \qquad (47)$$

In the inequalities (46) and (47) we replace the expression for a and b and neglecting terms of second degree in $\phi's$, we have finally

$$H + \bar{\epsilon} > 0 \; ; \; \bar{\nu} - 4\, M \geqslant \bar{\nu}^2 \left[(2\, T - H - D) \Big/ (H + \bar{\epsilon}) \right]^2$$

where

$$\left.\begin{aligned} \bar{\epsilon} &= J_g\,(1 - \epsilon_1) + D\epsilon_1 = J_g\;(D - J_g)\,\epsilon_1 \\ &\text{and} \\ \epsilon_1 &= \bar{\nu}^{2} \left[(\bar{\nu}^2 - 4\,M) + 2\,(2\,T - H - D)^2 \right] \\ &\div \left[(\bar{\nu}^2 - 4\,M)^2 + 4\,\bar{\nu}^2\,(2\,T - H - D)^2 \right] \\ &\sim \bar{\nu}^2 \Big/ (\bar{\nu}^2 - 4\,M), \left[if\,, \bar{\nu}^2 - 4\,M >> (2\,T - H - D) \right] \end{aligned}\right\} \quad (A)$$

and

$$\left.\begin{aligned} H + \bar{\epsilon} &= 0 \\ \bar{\nu}\,(2\,T - H - D) &= 0 \\ \bar{\nu}^2 - 4\,M &\geqslant 0 \end{aligned}\right\} \quad (B)$$

as our stability conditions, *i.e.* either (A) or (B) must be satisfied for the projectile to have dynamic stability.

We observe that the above conditions require that $\bar{\nu}^2 - 4M$ must be positive; ϵ_1 is positive and attains the minimum value unity for infinite spin ($\nu \to \infty$). If $(D - J_q)$ be positive which is satisfied over the upward branch of any trajectory and is true over the entire length of flat trajectories dealt with in spark-range work, the minimum value of ϵ_1, corresponds to the minimum of $\bar{\epsilon}$. Thus $Min.\ \epsilon_1 = 1$ and $Min.\ \bar{\epsilon} = D$ for flat trajectories; the stability conditions are

$$\left.\begin{aligned} H + D &> 0 \\ \bar{\nu}^2 - 4M &\geqslant \bar{\nu}^2 \left[2\,(T - H - D) \,/\, (H + D) \right]^2 \end{aligned}\right\} \quad (C)$$

where H and D are given by (30) and (32). These are precisely the conditions of stability laid down by McShane and Kelley[6]. If we choose the stability parameters as

$$S_1 = \phi_L + \bar{K}^{2}\,\phi_H - \phi_R$$

$$S_2 = 2\,\phi_L - 2\,\phi_R - 2m\,(d^2/A)\,\phi_J$$

$$S_3 = 2\,\bar{K}^{2}\,\phi_H + (2\,\phi_J - 2\phi_I\,)\,(md^2/A)$$

and

$$S = A^2\,\nu^2/(4\,B^2\,\bar{K}^{2}\,\phi_M)$$

(C) may be written as

$$1/S < S_2\,S_3/S_1^2\;,\; S_1 > 0$$

which also implies that $S > 1$. Regarding conditions (B), we define with Murphy the following conditions

$$\bar{v}^2 - 4M \geqslant 0 \quad i.e.\ S \ngtr 1 \tag{4)8}$$

as the gyroscopic stability conditions. Where the spin is zero, (48) gives

$$M < 0 \tag{49}$$

$$\phi_M \leqslant 0 \tag{50}$$

which happens if the centre of pressure is behind the centre of mass of the projectile. (50) is referred to as condition for static stability. For fined projectiles—(missiles), (50) is satisfied and they are statically stable, whence the following conclusions may be drawn.*

(1) Gyroscopic stability is sufficient for dynamic stability of statically stable projectiles

(2) Gyroscopic stability is necessary for dynamic stability.

(3) If the spin is zero, static stability is necessary for gyroscopic stability.

(4) For any spin, static stability is sufficient for gyroscopic stability.

(5) If $H + \bar{\epsilon} \geqslant 0$ and $\nu = 0$, static stability is sufficient for dynamic stability.

(6) If $H + \bar{\epsilon} \geqslant 0$ and $2T - H - D = 0$, gyroscopic stability is sufficient for dynamic stability.

PART II

Stability of steady conically yawing motion

The top analogy: Consider the yawing motion of a spinning shell. The undamped motion in yaw of the shell is nearly the same as that of the common top.

(In Fig. 1 $\vec{OP}$ is the axis of the projectile, OT is the direction of motion of its cm'O' Q is the point, usually known as centre of pressure, where the aerodynamic force $\vec{F}$ acts)

The air couple which tends to increase the yaw has its axis normal to the plane of yaw OPT (Fig.1) and is written as[5]

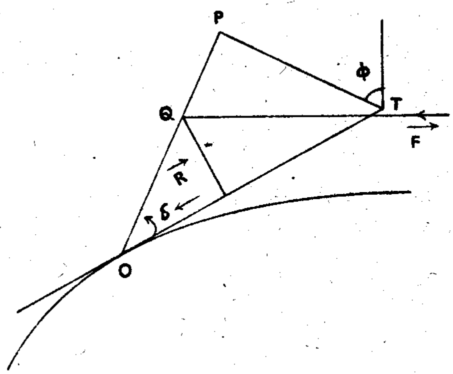

Fig. 1

$$M = \mu\,(\delta, v/a)\ \sin\delta \tag{51}$$

On comparison with (3) we have

$$\mu = \rho\, d^3\, v^2 f_M \tag{52}$$

If $\mu = const$, the figure axis OP of the shell executes the same type of angular oscillations about OT as a common top with the same dynamical specifications of the shell would if its point of support coincides with 'O' and centre of gravity with Q and the upsetting moment due to gravity for the top has the same value M of the shell throughout the motion.

*These are due to Murphy[1]

The condition $\mu = const.$ is satisfied to a very good approximation for most of the projectiles in use. This is because during initial motion, the velocity of the shell is nearly uniform and for small yawing motion as for stable shells in use the moment factor μ is fairly independent of δ and v/a. But if initial yawing motion is quite considerable as it happens in case of projectiles fired broadside from a moving aeroplane we may assume μ as an even function of δ given by

$$\mu(\delta) = \sum_{\lambda} \mu_{\lambda} \cos \lambda\delta \tag{53}$$

where μ_{λ}, $\lambda = 1,2$ are constants. The following convenient form of (53) valid upto $\delta = 35°$ has been used by Fowler and Lock[5]

$$\mu(\delta) = \frac{B\Omega^2}{4S}\left\{1 - 4\, qs\,(1 - \cos\delta)\right\} \tag{54}$$

The dimensionless constants S and q occuring in (54) are stability factors. It will appear from subsequent analysis and also it has been proved by Fowler and Lock that the motion of the shell axis with δ permanently zero, is stable or unstable according as $S > 1$ or $S < 1$. These conditions are similar to stability instability conditions of a sleeping top, whereas the motion in steady state of a common top is essentially stable, this is not always so for the shell especially when $q < 0$

Equation of yawing motion

If we described the angular motion of the shell-axis by δ, ϕ, Ψ †, the Eulerian angles, such motion is characterised by

$$\delta \int_0^t I\, dt = 0 \tag{55}$$

where the Kinetic potential L is given by $L = T - \{ - \int_0^{\delta} M d\,\delta \}$

$$\text{i.e.}\quad L = \tfrac{1}{2}\{B\delta'^2 + B\phi'^2 \sin^2\delta + A(\psi' + \phi'\cos\delta)^2\}$$
$$- \frac{B\Omega^2}{4S}\left\{\cos\delta - 4\, qS\,(\cos\delta - \tfrac{1}{2}\cos^2\delta)\right\} \tag{56}$$

T is the rotational $K.E$ of the shell and zero yaw position is treated as the position of zero potential energy.

The integrals of the Euler-Lagrange equation following from (55) for the cyclic coordinates ϕ and ψ give us

$$B\phi' \sin^2\delta + A(\psi' + \phi'\cos\delta)\cos\delta = const = BF$$

and

$$A(\psi' + \phi'\cos\delta) = const = B\Omega$$

From these two equations we have

$$\phi' \sin^2\delta + \Omega\cos\delta = F \tag{57}$$

Writing the reduced Kinetic potential

$$R = \tfrac{1}{2} B\Big[\delta'^2 + (F - \Omega\cos\delta)^2/\sin^2\delta$$
$$- \frac{\Omega^2}{2S}\left\{\cos\delta - 4\, qs\,(\cos\delta - \tfrac{1}{2}\cos^2\delta)\right\}\Big] + const. \tag{58}$$

† here ψ is not the same angle of drift as has been used previously

the motion equation in yaw is given by

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \delta'}\right) - \left(\frac{\partial R}{\partial \delta}\right) = 0$$

$$i.e.\ \delta'^2 \sin^2\delta + (F - \Omega \cos\delta)^2 + \Omega^2 \frac{\sin^2\delta}{2S}\left\{(1-4qs)\cos\delta + 2qs\cos^2\delta\right\}_{\delta}^{0} = E\sin^2\delta \qquad (59)$$

in the form studied previously by Fowler and Lock[5] and also by Kebbey[8]

Condition for steady conical yawing motion

For a steady conical yawing motion of the shell we must have

$$\frac{\partial R}{\partial \delta} = 0 \qquad (60)$$

This condition is of the same type as required for the steady precessional motion of a common top.

At the steady-state position of the shell if we suppose $\delta = \alpha$ and $\phi' = n\,\Omega$ where n is constant, we have from (60)

$$n^2 \cos\alpha + \frac{1}{4S}\{1-4qs(1-\cos\alpha)\} - n = 0 \qquad (61)$$

and from (57)

$$n \sin^2\alpha = \frac{F}{\Omega} - \cos\alpha \qquad (62)$$

Thus (61) and (62) are the conditions for steady precessional (conically yawing) motion of the shell. Writing $Z_\circ = \sin^2\frac{\alpha}{2}$, equation (61) reduces to

$$\frac{1}{S} - 4n(1-n) = 8Z_\circ(n^2+q) \qquad (63)$$

which is the condition due to Kebbey[8] † obtained by graphical methods.

Equations (61) and (62) give the relation between the constants n, α and Ω *i.e.* the values which the precessional angular velocity ϕ' and yaw δ assumes depending on the rate of spin of the shell (of given shape and size) round its axis of symmetry.

For the existence of a steady precessional motion, η as given by (61) must be real and therefore, the inequality

$$1 - \frac{\cos\alpha}{S}\{1-4qs(1-\cos\alpha)\} \geqslant 0 \qquad (64)$$

must be satisfied. When $q = 0$, (64) gives

$$s > \cos\alpha \qquad (65)$$

†Kebbey uses the constant c for n

which is the condition for the common top; (65) is a sufficient condition for (64) even when $q \geqslant 0$.

Stability of motion in the steady-state

Condition (60) implies that for steady precessional motion the potential energy of the reduced system, appearing in (58), must be stationary at $\delta = \alpha$, *i.e.* the steady state position. If we write the potential energy expression as

$$V(\delta) = \frac{B}{2}\left[\frac{(F - \Omega\ \cos\delta)^2}{\sin^2\delta} - \frac{\Omega^2}{2s}\left\{\cos\delta - 4\ qs\ (\cos\delta - \tfrac{1}{2}\cos^2\delta)\right\}\right] \tag{66}$$

excepting an additive constant term; at the position of steady state we have

$$\left(\frac{dV}{d\delta}\right)_\alpha = 0 \tag{67}$$

and for stability of motion in the steady state it is both necessary and sufficient that we have

$$\left(\frac{d^2 V}{d\,\delta^2}\right)_\alpha > 0 \tag{68}$$

The later condition is

$$B\,\Omega^2 \{ (1 - 2\,n\cos\alpha)^2 + (n^2 + q)\sin^2\alpha \} > 0 \tag{69}$$

when one makes use of the steady state conditions (61) and (62). When $q = 0$ (69) is essentially satisfied and we say the steady state motion of a common top is essentially stable. It is clearly seen from (69) that even when $q \geqslant 0$ the steady state motion of the shell is clearly stable. If however $q < 0$ and we write $q = -|q|$ condition (69) implies

$$|q| < n^2 + (\,cosec\,\alpha - 2n\cot\alpha\,)^2 \tag{70}$$

One can determine the stable nutational oscillations of the shell about its steady state position as follows: In (58) if we put $\delta = \alpha + x$ treating x to be small and expanding R in ascending powers of x upto second degree only, we have

$$R = \tfrac{1}{2}\,B\,x'^2 - \tfrac{1}{2}\left(\frac{d^2\,V}{d\,\delta^2}\right)x^2 + \ldots\ldots \tag{71}$$

since $\left(\dfrac{\partial V}{\partial \delta}\right)\alpha = 0$ when $\delta = \alpha$. Further $V(\alpha)$ being a constant is not included in (71) as this will have no contribution to the equation of motion.

$$\frac{d}{dt}\left(\frac{\partial R}{\partial x'}\right) - \frac{\partial R}{\partial x} = 0 \tag{72}$$

which immediately leads us to

$$x'' + \Omega^2 \{ (1 - 2\,n\cos\alpha)^2 + (n^2 + q)\sin^2\alpha \}\,x = 0 \tag{73}$$

due to (69), giving the period of stable nutational vibrations to be

$$\frac{2\pi}{\Omega}\left\{(1 - 2\,n\cos\alpha)^2 + (n^2 + q)\sin^2\alpha\right\}^{\frac{1}{2}} \tag{74}$$

It is a well known dynamical principle that the vibrations about steady state position of a dynamical system are infact the same as the vibrations about equilibrium position of the reduced system to which the problem is brought by ignoration of co-ordinates. Hence the steady state positions of the natural system characterised by differential equations following from (55) should correspond to the equilibrium position of the non-natural (reduced) system given by the differential equation (59). The closed phase trajectories of the system (59) will, therefore, correspond to the stable nutational oscillations of the shell axis about the steady state position. The phase trajectories given by (59) may be expressed as

$$y^2 = 4\,qZ^4 - \left(\frac{1}{s} + 4q\right) Z^3 - \left(H + 1 - \frac{1}{s}\right) Z^2 + (2h + H) - Zh^2 \qquad (75)$$

if we write in (59)

$$\frac{dZ}{d\,\Omega\,t} = y,\ Z = \sin^2\frac{\delta}{2}\ ,\ Z_o = \sin^2\frac{\delta_o}{2}$$

and

$$h = (1 - 2\,C)\,Z_o + 2\,CZ_o^{\,2} \qquad (76)$$

$$H = b^2 + \left(4\,C^2 - \frac{1}{\delta}\right)Z_o - 4\,(C^2 - q)\,Z_o^{\,2} \qquad (77)$$

using initial conditions at $t = 0$, $\delta = \delta_o$, $\frac{d\delta}{d\Omega t} = b$, $\frac{d\phi}{d\Omega t} = C$. They can be drawn in the phase plane (yz), noting that the dynamically possible motion[9] are for $(0 < z < I)$.

Stability or normal motion

During normal motion there is no yaw of the projectile. The stability condition of an equivalent (sleeping) top shows that this motion is stable or unstable according as $S > 1$ or $S \leqslant 1$. The same result, however, is true even when the moment coefficient of the shell is an even function of yaw as shown in (54). This has been proved by Fowler and Lock[5], by analysing the elliptic function solution of the equation (59). As we have shown here this result can also be obtained by examining the small oscillations of the shell axis about its normal position. To do so, we ignore the cyclic co-ordinate ψ and consider the reduced kinetic potential

$$R = \tfrac{1}{2}\,B\,(\delta'^2 + \sin^2\delta\,\phi'^2) + B\,\Omega\,\phi'\cos\delta$$
$$- \frac{B\Omega^2}{4s}\left\{\cos\delta - 4\,q\,s\,(\cos\delta - \tfrac{1}{2}\cos^2\delta)\right. \qquad (78)$$

and introduce small quantities

$$m = \sin\delta\cos\phi\ ,\ n = \sin\delta\sin\phi$$

in (78) and neglecting terms above second degree

in m, n, m' and n' we have

$$\begin{aligned} \delta'^2 + \phi'^2\sin^2\delta &= m'^2 + n'^2 \\ \phi'^2\sin^2\delta &= mn' - m'n \\ \cos\delta &= 1 - \tfrac{1}{2}\,(m^2 + n^2) \end{aligned} \qquad (79)$$

so that

$$R = \tfrac{1}{2}\,B\,(m'^2 + n'^2) - \tfrac{1}{2}\,B\,\Omega\,(mn' - m'n)$$
$$+ \frac{B\Omega^2}{8s}\,(l^2 + m^2) \qquad (80)$$

but for a constant term.

The Lagrangian equations now give

$$m'' + \Omega\, m' \quad \frac{\Omega^2}{4^S}\, l = 0 \tag{81}$$

$$m'' - \Omega\, n' - \frac{\Omega^2}{4^S}\, m = 0$$

which are the same equations as for the equivalent common top. As usual the oscillation characterised by (81) are stable or unstable according as $S > 1$ or $S \leqslant 1$.

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