PROPAGATION OF MAGNETOHYDRODYNAMIC SHOCK WAVES

T. D. VARMA & D. P. BATRA

Defence Science Laboratory, Delhi

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The propagation of magnetohydrodynamic shock waves in a self gravitating gas sphere has been studied. The shock is supposed to be generated as a result of sudden release of energy at the centre of symmetry. A similarity solution for the flow is developed. Numerical results for three different values of Mach-number given by $M^2=200, 100$ and 10 are obtained. The magnetic field affects the boundary conditions at the shock as well as the flow behind it. For a fixed M, the particle velocity, density and gas pressure decrease as the initial magntic field increases. In the flow field behind the shock, there exists a transition region across which the effect of magnetic field is reversed.

The study of propagation of shock waves in spherical gas models of astrophysical interest has been of considerable importance in explaining certain phenomenon such as novae and supernovae bursts. According to observations, such flares result from unsteady motion of large masses of gas as a result of sudden release of energy in the star. The unsteady motion of the gas, taking Newtonian gravitation into account, has been studied by Sedov[1]. The sudden release of energy at the centre of the star gives rise to an outward going shock wave. The impact of the shock is sufficient to eject matter from the periphery of the star with speed exceeding the velocity of escape from the gravitational field of the entire configuration. Carrus *et al.*[2] and Kopal[3] in a series of three papers have investigated the same problem by a purely numerical method.

In this paper, we have studied the propagation of a magnetohydrodynamic shock in a self gravitating steller model. The magnetic field is transverse to the direction of shock propagation. The shock is produced at the centre of model as a result of liberation of a finite amount of energy. The explosion is assumed instantaneous, an assumption which makes the total energy of the configuration independent of time.

FORMULATION OF THE PROBLEM

We consider a spherical model and assume the flow to be in the radial direction only. Consequently, all the flow quantities are functions of radial distance r and time t only. The gas is assumed to be ideally conducting; the effect of heat conduction, viscosity etc. are neglected. An initial magnetic field B_o in the θ direction only is supposed to be present in the medium. The flow is governed by the usual hydrodynamic equations with some modifications to include gravitation and magnetic field.

The equation of continuity is expressed as

$$\partial \rho/\partial t + u\, \partial \rho/\partial r + \rho\, (\partial u/\partial r + 2u/r) = 0 \qquad (1)$$

The momentum equation gives

$$\partial u/\partial t + u.\partial u/\partial t + 1/\rho.\, \partial/\partial r\, (p + B^2/2\mu) + B^2/\mu.\rho r + mG/r^2 = 0 \qquad (2)$$

where G is the gravitational constant and μ the magnetic permeability of the free space. p, ρ , u and $m\,(r, t)$ denote gas pressure, density, particle velocity and mass at any time t in a sphere of radius r respectively.

The equation determining $m\ (r, t)$ is

$$\partial m/\partial r = 4\pi\ \rho r^2 \tag{3}$$

The energy equation, neglecting all the dissipative mechanisms, is

$$(\partial/\partial t + u\ .\ \partial/\partial r)\ (p/\rho r) = 0 \tag{4}$$

Finally, we have the induction equation for magnetic field expressed as

$$\partial B/\partial t + u.\partial B/\partial r + B\ (\partial u/\partial r + u/r\ ) = 0 \tag{5}$$

The system of equations (1—5) are to be solved subject to the Rankine–Hugoniot conditions discussed under boundary equations.

EQUILIBRIUM CONDITIONS

In the equilibrium case the system of equations (1—5) is reduced to the two equations

$$\partial/\partial r\ (p_o + B_o^2/2\mu) + B_o^2/\mu r + Gm_o\rho_o/r^2 = 0 \tag{6}$$

and

$$\partial m_o/\partial r = 4\pi\ \rho_o\ r^2 \tag{7}$$

The initial density ρ_o of the medium is assumed to vary as certain inverse power of r i.e., $\rho_o = Ar^{-\omega}$. Substituting in (6) and (7) we get

$$\left.\begin{aligned} m_o &= 4\pi A/(3-\omega)\ r^{3-\omega} \\ p_o &= K_1\ A^2\ G\ r^{2-2\omega} \\ B^2{}_o/2\mu &= K_2\ A^2\ G\ r^{2-2\omega} \end{aligned}\right\} \tag{8}$$

where K_1 and K_2 are abstract constants related by the equation

$$(\omega - 1)\ K_1 + (\omega - 2)\ K_2 - 2\pi/(3 - \omega) = 0 \tag{9}$$

(8) shows that the centre of symmetry is a singular point at which the density, pressure etc. become infinite. In actual conditions they are never infinite but may be very high. However, neglecting the origin, (8) gives the variation of physical quantities in the equilibrium state. The above family of solutions depends on a dimensional constant A and a characteristic parameter ω, in addition to gravitational constant G.

From (8) it is evident that the mass will be real and increasing function of r only if $\omega < 3$. Pressure and magnetic field will be decreasing function of r when $\omega > 1$. Hence the limit of variation for ω is $3 > \omega > 1$. The value of ω for this problem, is determined from the requirement of constancy of total energy of the configuration.

SIMILARITY TRANSFORMATIONS

We have assumed that the gas motion starts at time $t = 0$ as a result of liberation of energy E_o. The law of energy liberation is determined by the three dimensional quantities t, A and G i.e.,

$$E_o = \alpha_1\ G^{(5/\omega - 1)}\ A^{5/\omega}\ t^{2\ (5 - 2\omega)/\omega} \tag{10}$$

where $(A) = M.\ L^{\omega - 3}$, $(G) = M^{-1}\ L^3\ T^{-2}$

The total energy will be time independent if $\omega = 5/2$. Then (10) becomes

$$E_0 = \alpha_1 G A^2 \tag{11}$$

The field of disturbed motion is determined by the system of dimensional parameters A, G, r and t. The gas motion is self similar and the laws of motion can be expressed as

$$\left.\begin{aligned} u &= (r/t)\,U, \quad \rho = R/G t^2 \\ p &= (r^2/Gt^4)\,P, \quad m = (r^3/Gt^4)\,M, \\ B^2/2\mu &= (r^2/Gt^4).\,\beta \end{aligned}\right\} \tag{12}$$

where U, R, P, β and M are non-dimensional quantities which are function of η only given by

$$\eta = r/(\alpha_2 A\, G)^{1/\omega}\; t^{2/\omega} \tag{13}$$

where α_2 is an abstract constant. Using (12) and (13) the system of equations (1—5) is reduced to

$$\left.\begin{aligned} \eta\,[(U-\delta)\ \ R'/R + U'\,] &= 2 - 3U \\ \eta\,[(U-\delta)\,U' + (P^1 + \beta')/R\,] &= U - U^2 - M - 2(P+2\beta)/R \\ \eta M' &= 4\pi R - 3M \\ \eta\,[\,(U-\delta)\ \ P'/P + \gamma U'\,] &= 4 - (3\gamma+2)\,U \\ \eta\,[\,(U-\delta)\ \ (\beta'/2\beta) + U'\,] &= 2 - 3U \end{aligned}\right\} \tag{14}$$

where dash denotes differentiation *w.r.t.* η and $\delta = 2/\omega$. The partial differential equations (1—5) are thus reduced to ordinary differential equations in η only.

BOUNDARY CONDITIONS

In terms of the physical quantities, the boundary conditions at the shock are

$$\left.\begin{aligned} \rho_s\,(C - u_s) &= \rho_o c \\ B_s\,(C - u_s) &= B_o c. \\ p_s - p_o + 1/2\mu\,(B_s^{\,2} - B_o^{\,2}) &= \rho_o c.u_s \\ \gamma p_s/(\gamma-1)\rho_s + B_s^{\,2}/\mu\rho_s + 1/2\,(C - u_s)^2 &= \gamma p_o/(\gamma-1)\rho_o + B_o^{\,2}/\mu\rho_o + 1/2C^2 \\ m_s &= m_o \end{aligned}\right\} \tag{15}$$

where the subscript s denotes the values of physical quantities just behind the shock and C, the shock velocity.

The shock radius r_1 is determined by three dimensional quantities G, A, and t. Thus

$$r_1 = \eta_1\,(\alpha_2 AG)^{1/\omega}\, t^{2/\omega}, \qquad \eta = \eta_1 = \text{Constant} \tag{16}$$

The constant value of $\eta = \eta_1$ at the shock wave can be taken as unity. Thus we have $\eta_1 = 1$, $\eta = r/r_1$

The shock velocity C is given by

$$C = d/dt = 2r_1/\omega t \tag{17}$$

We put

$$\left.\begin{aligned} \lambda_o^2 &= a_o^2/c^2 = (\gamma k_1/\alpha_2)(\omega/2)^2 \\ \beta_o^2 &= b_o^2/c^2 = (2k_2/\alpha_2)(\omega/2)^2 \end{aligned}\right\} \tag{18}$$

where a_o and b_o represent the sound velocity and the velocity of Alfven waves in the undisturbed medium respectively.

After transformation to non-dimensional form, the equations (15) yield

$$\alpha_2 R = \{1 - (\omega/2)\, U\}^{-1} \tag{19}$$

$$\alpha_2 \beta = 2/\omega^2\, \beta_o^2\, (\alpha_2 R)^2 \tag{20}$$

$$\alpha_2 P = (2/\omega)^2 \left[(\omega/2)\, U + (\lambda_o^2/\gamma) - (\beta_o^2/2)\, \{(\alpha_2 R)^2 - 1\}\right] \tag{21}$$

$$[1 - \lambda_o^2 (\gamma + 1)/2\ (\omega/2)]\, U\, [1 - (\omega/2)\, U] = \beta_o^2\, [(1 - (\gamma/2)(\omega/2)\, U] \tag{22}$$

$$\alpha_2 M = 4\pi/(3 - \omega) \tag{23}$$

The constant α_2, occuring above, is related to λ_o^2 and β_o^2 by the formula

$$\alpha_2 = \pi\gamma\omega^2/(3 - \omega)\, [2\,(\omega - 1)\, \lambda_o^2 + \gamma\,(\omega - 2)\, \beta_o^2] \tag{24}$$

The system of equations (19—23) define two parameter family of shock waves, if the quantities λ_o^2 and β_o^2 are considered as parameters. For a given pair of values of λ^2_o and β_o^2, U is first determined as a root of quadratic equation (22). We then get R from (19), β from (20) and P from (21). Thus the values of all non-dimensional quantities just behind the shock front are determined.

REDUCTION OF EQUATIONS

The system of equations (14) consists of five ordinary differential equations in R, P, U, M and β. Three algebraic integrals of the equations are obtained. Thus

$$M = 2\pi\, R\, (4 - 5U) \tag{25}$$

(Mass integral)

$$P/R^{\gamma} = \alpha_3 \left[R\,(4 - 5U)\right]^{5\gamma - 6} . \eta^{15\gamma - 20} \tag{26}$$

(Entropy integral)

$$\beta = \alpha_4 R^2 \tag{27}$$

(Freezing-in integral)

With the help of these, the order of system (14) can be reduced by three. We thus get the following two differential equations in R and U.

$$5\, RU' - (4 - 5U)\, R' = (10 - 15U)\, R/\eta \tag{28}$$

$$\eta \left[10\alpha_4 R' + (4 - 5U)U' \left\{25\alpha_3\gamma R^{6\gamma-7}(4-5U)^{5\gamma-8}\eta^{15\gamma-20} - 1\right\}\right]$$

$$= 5U(1-U) - 10R(4\pi + 2\alpha_4 - 5\pi U) + \alpha_3(60 - 75\gamma U)(4-5U)^{5\gamma-7} R^{6\gamma-7}\eta^{5\gamma-20} \tag{29}$$

Equations (26) and (29) for $\gamma = 4/3$ take the form

$$P = \alpha_3 R^2 (4 - 5U)^{2/3} \tag{30}$$

$$\eta\left[30\alpha_4 R' + (4-5U)U'\left\{100\alpha_3 R(4-5U)^{-4/3} - 3\right\}\right]$$

$$= 15U(1-U) - 30R(4\pi + 2\alpha_4 - 5\pi U) + 60\alpha_3(3-5U)(4-5U)^{-1/3} R \quad (31)$$

Equations (25), (27), (28), (30) and (31) represent the complete solution.

NUMERICAL INTEGRATION

No further analytical solution of the equations (29) and (31) was possible. In order to investigate the field of flow, recourse must be had to numerical intergration. Numerical integration for three different values of Mach-number was performed on IBM digital computor. $1/M^2$ was denoted by q and three values 0·005, 0·01 and 0·1 were selected for q. For each Mach-number three different cases, as mentioned below, were considered.

Case 1 $\lambda_o^2 = q$, $\beta_o^2 = 0$

i.e., the case of explosion without a magnetic field.

Case 2 $\lambda_o^2 = q/2$, $\beta_o^2 = q/2$

The gas pressure and magnetic pressure are comparable.

Case 3 $\lambda_o^2 = 0$, $\beta_o^2 = q$

The initial magnetic pressure is assumed to be much larger than the hydrostatic gas pressure. The gas pressure is ignored. In all nine computations were performed.

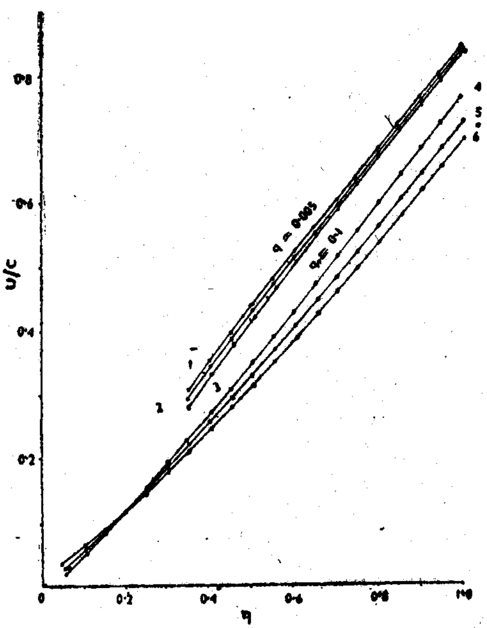

FIG. 1—Particle velocity as a function of distance from the centre of symmetry.

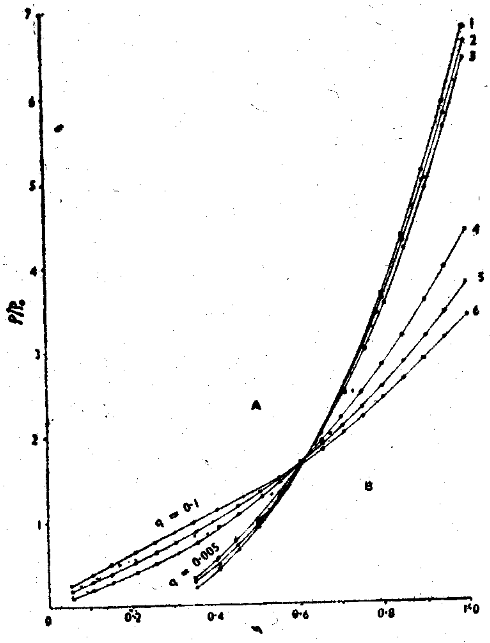

FIG. 2—Density of gas as a function of distance from the centre of symmetry.

First of all the values of non-dimensional quantites U, R, P, M and β were determined at the shock for the nine sets of values of λ_o^2 and β_o^2 from equations (19) to (23). These were then substituted in (27) and (30) to get the corresponding values of constants α_3 and α_4.

The differential equations (28) and (31) in U and R were solved by Modified Euler's method. The quantities M, β and P were then evaluated for each value of η. The computation was started from $\eta = 1$ *i. e.* the shock and was carried out towards the origin with initial integration $d\eta = 0\cdot01$, which was later changed to $0\cdot001$ and $0\cdot0001$ when the convergence in integration could not be attained with given interval. The values of computation for $q = 0\cdot005$ and $q = 0\cdot1$ are plotted in Fig. 1—5.

DISCUSSION

Fig. 1—5 show the variation of particle velocity U, density ρ, pressure P, magnetic pressure $\beta^2/2\mu$ and mass m as a function of distance from the centre of symmetry. Curves 1, 2, 3 are for $q = 0\cdot005$ and 4, 5, 6 for $q = 0\cdot1$. Curves 1,4 give variation for case 1 *i.e.*, $\lambda_o^2 = q$, $\beta_o^2 = 0$; curves 2,5 for Case 2 *i.e.*, $\lambda_o^2 = \beta_o^2 = q/2$ and curves 3,6 for Case 3 *i.e.*, $\lambda_o^2 = 0$, $\beta_o^2 = q$. Our results for Case 1 go over to those of Sedov [1].

From the analysis of the results it is quite clear that the presence of a magnetic field ahead of the shock has an important effect on conditions at the shock and everywhere behind it.

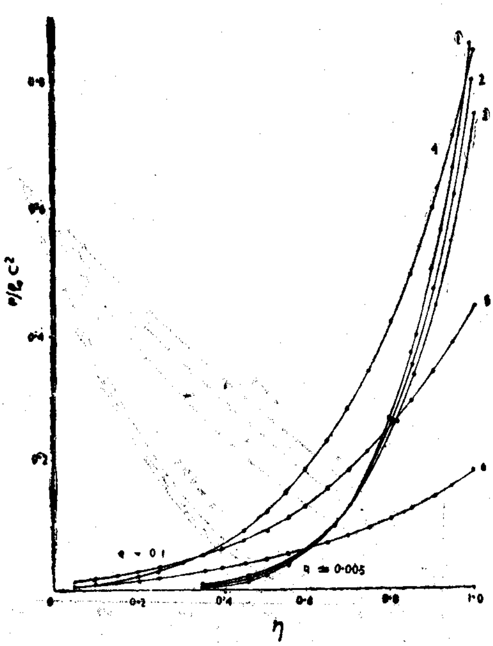

FIG. 3—Pressure of gas as a function of distance from the centre of symmetry.

The particle velocity decreases as the centre of symmetry is approached. There are two factors which are to be taken into account *viz.* Mach-number and magnetic field. Particle velocity is a direct function of Mach-number. For $q = 0\cdot005$ and $0\cdot01$, as the magnetic field is increased, the particle velocity decreases. For $q = 0\cdot1$, the effect is of slightly different nature. As the magnetic field is increased (*i.e.*, β_o^2 increases), the particle velocity at the shock and upto some value of η (between $0\cdot15$ to $0\cdot2$) in the flow field decreases, beyond which the effect is gradually reversed. In the transition region the velocity is independent of magnetic field strength.

Shock compression (ρ/ρ_o) decreases from its strong shock value of $(\gamma+1)/(\gamma-1)$ ($=7$ for $\gamma = 4/3$) as the Mach-number is decreased. For a fixed Mach-number, the compression at the shock decreases with magnetic field. The curves in Fig. 2 reflect that the effect of magnetic field itself is a function of the distance from the centre of symmetry. To obtain a clear picture, the entire flow region can be divided into two parts separated by a narrow transition region. In region B, compression decreases with magnetic field whereas in region A the reverse is true.

The value of gas pressure decreases as the Mach-number is decreased. For $q = 0 \cdot 005$ and $0 \cdot 01$, the flow region is again divided into two parts. The transition region lies between $\eta = 0 \cdot 6$ to $0 \cdot 7$. Initially the pressure decreases as the magnetic field increases till at the transition point the effect is reversed. It is also clear from Fig. 3 that the effect of the magnetic field is more prominent for lower Mach-numbers.

Fig. 4 shows the magnetic pressure as a function of η. For lower Mach-number the magnetic pressure is higher. For a fixed Mach-number, as the initial magnetic field (β_o^2) is increased, the magnetic pressure also increases. There is no transition region in this case.

The mass of the gas is plotted in Fig 5. It decreases as the origin is approached, the rate of decrease being higher for higher Mach-number. The curves start from $m/m_o = 1$ at the shock. The magnetic field increases the value at all the points in the flow field.

From the analysis it is clear that, for $q = 0 \cdot 005$ and $0 \cdot 01$, the flow does not extend to the centre of symmetry. A spherical vacuum of radius r^* forms at the centre of symmetry in which $r^* = \eta^* r_1$ where the constant η^* depends only on q and equals the value of parameter η at the interior boundary where the pressure and density vanish. For $q = 0 \cdot 1$ the disturbance created by the explosion occupies the whole interior of the shock wave. The flow extends to the centre of symmetry.

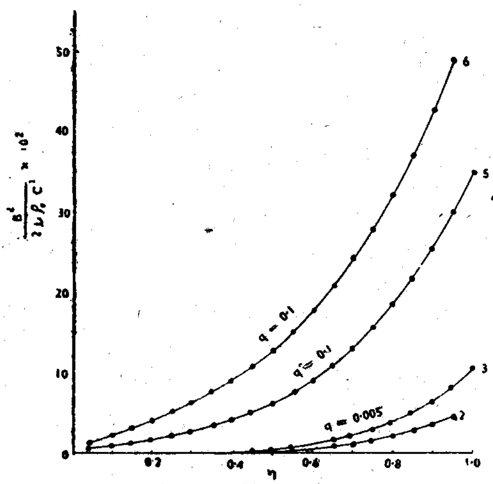

FIG. 4—Magnetic pressure $B^2/2\mu$ in units of $\rho_o C^2$ as a function of distance from the centre of symmetry.

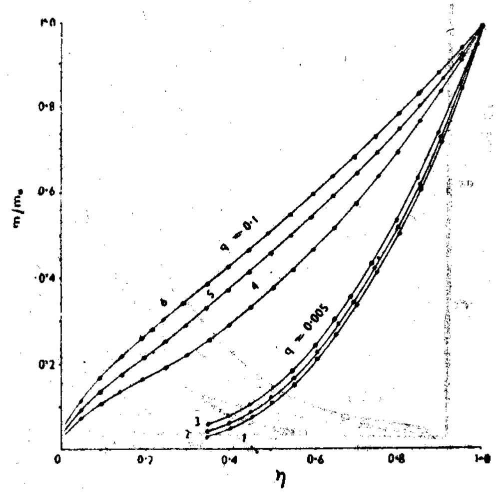

FIG. 5—Mass of gas m/m_o as a function of distance from the centre of symmetry.

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