ESTIMATION OF THE PARAMETERS OF A POISSON-RECTANGULAR DISTRIBUTION

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The application of Poisson-Rectangular distribution in an industrial sampling problem, when the process mean is subjected to fluctuations, is indicated. Moment estimators of the parameters involved are discussed in detail. A problem of misclassification in the inspection of defects on finished articles and manufactured products is also included.

The authors[1] have already obtained a Poisson-Rectangular distribution (P-R distribution) which arises when the parameter involved in a Poisson distribution is itself a random variable distributed uniformly over a certain range $[\alpha, \beta]$. The applications of such parametric variations in the field of accident statistics have already appeared.[2,3] The object of this paper is to consider the moment method of estimation of the parameters involved in P-R distribution with regard to an industrial application. To visualise one, let us consider batches of reels of enamelled wire, being tested for pinholes in the enamel. The test is on a reel over a standard length drawn through a bath of electrically conducting liquid so that the insulation breaks down at every pinhole in the enamel. We first consider the method of sampling. Several miles of wire are made at one time and it is afterwards, divided into reels. By the time these reels reach the testing department they would be thoroughly mixed. Tests on successive lengths of one wire might show little significant variation compared with those from one reel to another and it is frequently assumed that within one wire, the number of holes in standard lengths follow Poisson distribution with the parameter λ. There is, however, an inevitable and continuous change in conditions of manufacture leading to fluctuations in this parameter. These fluctuations may be negligible over the standard lengths in the same reel but this is not so if tests are performed on standard length each from a set of n reals included in the sample. If these random variations in the parameter, that are not negligible for such a method of sampling, are assumed to be governed by a uniform distribution over the range $[\alpha, \beta]$, the resulting sample would be the one from a P-R distribution considered in this paper.

Further, the estimation of the parameters based under the assumption of misclassification of defects in manufactured articles and finished products analogous to that of Cochen[4] is also considered.

P-R DISTRIBUTION

We consider the probability distribution

$$f\ (x) = \frac{e^{-\lambda}\ \lambda^{x}}{x\,!}\ , \qquad \lambda > 0\ ;\ x = 0,\ 1,\ 2, \ldots\ldots \tag{1}$$

where λ is itself a random variable with a probability density.

$$g(\lambda) = \left(\frac{1}{\beta-\alpha}\right) \qquad 0 < \alpha < \lambda < \beta \tag{2}$$
$$= 0 \text{ otherwise}$$

The over all probability distribution of x is obtained by multiplying (1)and (2) and integrating over the range involved. Thus P-R distribution is given by the probability distribution

$$h(x) = \left\{\frac{1}{(\beta-\alpha)}\right\} \int_{\alpha}^{\beta} \left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right) d\lambda \tag{3}$$
$$= \left[\frac{1}{(\beta-\alpha)(x+1)!}\right] \left\{\beta^{x+1} \phi(x+1; x+2; -\beta) - \alpha^{x+1} \phi(x+1; x+2; -\alpha)\right\}$$
$$x = 0, 1, 2, \ldots\ldots$$
$$0 < \alpha < \beta$$

Obtained by expressing the integral as a difference of two integrals and then using the following result due to Erdelyi[5]

$$\int_{0}^{z} e^{-t} t^{a-1} dt = \frac{1}{a} z^{a} \phi(a; a+1; -z) \tag{4}$$

where the confluent hypergeometric function #is defined as

$$\phi(a;b;z) = \sum^{\infty} \left\{\frac{(a)_n}{(b)_n}\right\} \left\{\frac{z^n}{n!}\right\} \tag{5}$$

the series being absolutely convergent for all values of a, b and z, real or complex, excluding $b=0, -1, \ldots\ldots\ldots\ldots\ldots$

$$(a)_n = a(a+1) \ldots\ldots\ldots\ldots\ldots (a+n+1) \text{ with } (a)_0 \equiv 1 \tag{6}$$

The special case, $\alpha=0$ lead to a neat form

$$h(x) = \left\{\frac{\beta^{x}}{(x+1)!}\right\} \phi(x+1; x+2; -\beta) \tag{7}$$
$$\beta > 0; \; x = 0, 1, 2, \ldots$$

which will be used later.

The moment generating function has been shown to be[1]

$$M(t) = \left\{\frac{1}{(\beta-\alpha)(e^{t}-1)}\right\} \left[\exp\left\{\beta(e^{t}-1)\right\} - \exp\left\{\alpha(e^{t}-1)\right\}\right] \tag{8}$$

from which the first four moments about the origin are evaluated as

$$\left.\begin{aligned} \mu'_1 &= (\alpha+\beta)/2; \; \mu'_2 = \{(\alpha+\beta)/2\} + (\alpha^2+\beta^2+\alpha\beta)/3 \\ \mu'_3 &= \{(\alpha+\beta)/2\} + (\alpha^2+\beta^2+\alpha\beta) + (\beta^3+\beta^2\alpha+\beta\alpha^2+\alpha^3)/4 \\ \mu'_4 &= \{(\alpha+\beta)/2\} + 7/3\,(\alpha^2+\beta^2+\alpha\beta) + 3/2\,(\beta^3+\beta^2\alpha+\beta\alpha^2+\alpha^3) \\ &\quad + (\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)/5 \end{aligned}\right\} \tag{9}$$

MOMENT ESTIMATES OF THE PARAMETERS OF P-R DISTRIBUTIOS

Let us now consider the estimation of the parameters by the method of moments based on a random sample ($x_1, x_2, \ldots, x_n$) from the distribution specified by (3). Denoting the sample moments about origin by

$$m'_r = \left(\frac{1}{n}\right) \sum_{i=1}^{n} x_i^r \tag{10}$$

and equating the first two sample moments with those of the population at (9), the moment estimates, if they exist, are given by

$$\left.\begin{aligned} \hat{\beta} &= m'_1 + \{3\,(m'_2 - m'^2_1 - m'_1)\}^{\frac{1}{2}} \\ \hat{\alpha} &= m'_1 - \{3\,(m'_2 - m'^2_1 - m'_1)\}^{\frac{1}{2}} \end{aligned}\right\} \tag{11}$$

Evidently it is possible that the expressions given in (11)may turn out to be imaginary as for example this is the case if the sample mean exceeds the sample variance. However, with increasing sample size the probability of such roots is negligibly small. This problem has been studied by Paul Rider[6]. The calculations of variances, in large samples, is achieved by using the following formula due to Cramer.[7]

$$\begin{aligned} V(\hat{\alpha}) = {} & \left(\frac{\partial \hat{\alpha}}{\partial m'_1}\right)^2 \mu_2(m'_1) + 2\left(\frac{\partial \hat{\alpha}}{\partial m'_1}\right)\left(\frac{\partial \hat{\alpha}}{\partial m'_2}\right) \mu_{11}(m'_1, m'_2) \\ & + \left(\frac{\partial \hat{\alpha}}{\partial m'_2}\right)^2 \mu_2(m'_2) \end{aligned} \tag{12}$$

where $\mu_2(m'_1)$, $\mu_{11}(m'_1, m'_2)$ and $\mu_2(m'_2)$ are second order moments of sample moments m'_1 and m'_2 and the partial derivatives are to be evaluated at the points $m'_1 = \mu'_1$ and $m'_1 = \mu'_2$. The values of these partial derivations are found to be

$$\left.\begin{aligned} \frac{\partial \hat{\alpha}}{\partial m'_1} &= \frac{(2\alpha + 4\beta + 3)}{(\beta - \alpha)} \\ \frac{\partial \hat{\alpha}}{\partial m'_2} &= \frac{-3}{(\beta - \alpha)} \end{aligned}\right\} \tag{13}$$

Substituting the values of μ'_k (k=1, 2, 3, 4) from (9) in the following formula due to Kendall[8], we get.

$$\mu_{11}\left(m'_r, m'_s\right) = \frac{1}{n}\left(\mu'_{r+s} - \mu'_r\,\mu'_s\right) \tag{14}$$

we obtain the second order moments of sample moments. Substituting these and the values of the partial derivatives given by (13) in (12), we obtain. after a laborious simplification, the following expression for the variance of the estitnate of $\hat{a}$.

moments about origin with those of the sample. Thus the moment estimators, if they exist, are given by

$$\hat{\beta} = \{3\ (m'_2 - m'_1)\}^{\frac{1}{2}}$$
$$\hat{\theta} = \left[1 - \frac{2\ m'_1}{\{3(m'_2 - m'_1)\}^{\frac{1}{2}}}\right] \div \phi\left[2; 3; -\{3\ (m'_2 - m'_1)\}^{\frac{1}{2}}\right] \qquad (20)$$

The large sample variances of these estimators are obtained in the same manner as before. Using the differential formula for the confluent hypergeometric function[5], the partial derivatives evaluated at the points $m'_1 = \mu'_1$ and $m'_2 = \mu'_2$ are found to be

$$\left.\begin{aligned}
&\frac{\partial\hat{\beta}}{\partial m'_1} = \frac{-3}{2\beta}\ ; \qquad \frac{\partial\hat{\beta}}{\partial m'_2} = \frac{3}{2\beta} \\
&\frac{\partial\hat{\theta}}{\partial m'_1} = -\left\{4\ \beta + 2\ \beta\theta\ \phi(3; 4; -\beta) + 3 - 3\ \theta\ \phi\ (2; 3; -\beta)\right\} \div 2\ \beta^2\ \phi(2; 3; -\beta) \\
&\frac{\partial\hat{\theta}}{\partial m'_2} = \left\{2 \cdot \beta\theta\ \phi(3; 4; -\beta) + 3 - 3\ \theta\ \phi\ (2; 3; -\beta)\right\} \div 2\ \beta^2\ \varphi(2; 3; -\beta)
\end{aligned}\right\} \qquad (21)$$

Evaluating the second order sample moments with the help of (14) and (19) and substituting these and the partial derivatives evaluated above the Cramer's formula, me obtain, after considerable simplification, the following expressions for asymptotic variance of these estimates

$$\left.\begin{aligned}
V\left(\hat{\beta}\right) &= \frac{1}{20n}\left[30 + 45\ \beta + 4\ \beta^2\right] \\
V\ (\hat{\theta}) &= \Big[6\beta(1 - 2\ \theta\phi_o + \theta^2\ \phi_o^2) \\
&\quad + \beta^2\ (1 + 8\ \theta\phi_1 - 10\ \theta\phi_o - 8\ \theta^2\ \phi_o\ \phi_1 + 9\ \theta^2\ \phi^2_o) \\
&\quad + \beta^3\left(\frac{2}{15} + \frac{8}{3}\theta^2\ \phi_1^2 + \frac{4}{3}\theta\ \phi_1 + \frac{22}{5}\theta\ \phi_o\right. \\
&\qquad \left. - 12\ \theta^2\ \phi_o\ \phi_1 + \frac{4}{5}\theta^2\ \phi^2_o\right) \\
&\quad + \beta^4\left(4\ \theta^2\ \phi_1^2 - \frac{4}{15}\theta\ \phi_1 - \frac{56}{15}\theta^2\ \phi_o\ \phi_1\right) + \frac{16}{45}\beta^5\ \theta^2\ \phi_1^2\Big] \div 4n\beta^3\phi^2_o
\end{aligned}\right\}$$

where φ_0 and φ_1 have been used to denote $\varphi\ (2; 3; -\beta)$ and $\phi\ (3; 4; -\beta)$ respectively for the sake of brevity.

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