ON THE CONSTITUTIVE RELATION FOR A CLASS OF REINER-RIVLIN FLUIDS

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For a class of Reiner-Rivlin fluids characterised by the relation $T = \mu_0 I + \mu_1 D + \mu_2 D^2 \ldots \ldots + \mu_n D^n$ where T, D and I are the stress, rats of strain and unit matrices respectively, restrictions on the phenomenological coefficients $\mu_0, \mu_1, \mu_2, \mu_n \ldots$ when they (i) are constants and (*ii*) accept power-series expansions, have been obtained so that the dissipation function is non-negative for all possible rates of deformation.

Eringen[1] has established that *(n)* there exists no incompressible fluid of second order and *(h)* the most general second order compressible fluid is a quasi-linear fluid. These results are based on the consideration that the dissipation function is non-negative throughout for all possible strain-rates.

Result (a) is very significant as it implies that incompressible Reiner-Rivlin fluids with constant coefficients of viscosity and cross viscosity do not exist., This may invalidate a large number of papers dealing with these fluids and their results have to be examined critically from this point of view.

In the present paper it is proposed to find out the restrictions on the material constants occurring in the constitutive relation for a class of Reiner-Rivlin fluids so that the dissipation function is non-negative for all possible strain-rates. These are characterised by the relation

$$T = \mu_0 I + \mu_1 D + \mu_2 D^2 + \ldots\ldots\ldots \mu_n D^n, \tag{1}$$

where T, D and I are respectively the stress, rate of strain and unit matrices. $\mu_0, \mu_1, \ldots \mu_n$ are the material constants. We discuss the problem in two cases (i) when $\mu_1, \mu_2, \ldots \mu_n$ are constants and (ii) when they accept power-series expansions. The method used in case (ii) has been illustrated by considering a tenth order fluid.

DISSIPATION FUNCTION FOR A CLASS OF REINER-RIVLIN FLUIDS

Consider the Reiner-Rivlin fluids characterised by constitutive equation (1). The dissipation function is given by the trace of the matrix

$$\mu_0 D + \mu_1 D^2 + \mu_2 D^3 + \ldots\ldots\ldots + \mu_n D^{n+1} \tag{2}$$

i.e. the dissipation function is given by

$$\phi = \mu_0 e_{ii} + \mu_1 e_{ij} e_{ji} + \mu_2 e_{ij} e_{jk} e_{ki} + \ldots\ldots (n \text{ terms}) \tag{3}$$

$$= \mu_0 I^{(1)} + \mu_1 I^{(2)} + \ldots\ldots\ldots + \mu_n I^{(n+1)} \text{ (say)}, \tag{4}$$

where $I^{(1)}, I^{(2)}, \ldots\ldots, I^{(n)}$ are the traces of the matrices D, $D^2, \ldots\ldots\ldots\ldots D^{(n)}$ ϕ is scalar and can be expressed in terms of the three invariants I_1, I_2, I_3 of the tensor e_{ij} thus

$$I^{(1)} = e_{ii} = I_1 \tag{5}$$

$$I^{(2)} = e_{ij}\, e_{ji} = I_1^2 - 2I_2 \tag{6}$$

From Cayley-Hamilton theorem

$$e_{ij}\, e_{jk}\, e_{kl} - I_1\, e_{ij}\, e_{jl} + I_2\, e_{il} - I_3\, \delta_{il} = 0 \tag{7}$$

Contracting the suffixes i and l, we get

$$I^{(3)} - I_1\, I^{(2)} + I_2\, I^{(1)} - 3I_3 = 0 \tag{8}$$

i.e.

$$I^{(3)} = I_1^3 - 3I_1\, I_2 + 3I_3 \tag{9}$$

Multiplying *(7)* by e_{lm} and contracting the suffixes i and m and simplifying we get

$$I^{(4)} = I_1^4 - 4I_1^2\, I_2 + 2I_2^2 + 4I_1\, I_3 \tag{10}$$

$$I^{(5)} = I_1^5 - 5I_1^3\, I_2 + 5I_1\, I_2^2 + 5I_1^2\, I_3 - 5I_2\, I_3 \tag{11}$$

Similarly all $I^{(r)}$ $(r = 2, 3, \ldots\ldots\ldots n)$ can be expressed in terms of I_1, I_2, I_3.

To obtain a general formula for $I^{(n)}$, we note that it satisfies the difference equation

$$I^{(n)} - I_1\, I^{(n-1)} + I_2\, I^{(n-2)} - I_3\, I^{(n-3)} = 0 \tag{12}$$

so that a general expression for $I^{(n)}$ is

$$I^{(n)} = A_1\, x_1^{\,n} + A_2\, x_2^{\,n} + A_3\, x_3^{\,n} \quad , \tag{13}$$

where A_1, A_2, A_3 are constants independent of n; x_1, x_2, x_3 are the characteristic roots of the matrix D *i.e.* the roots of the equation

$$x^3 - I_1\, x^2 + I_2\, x - I_3 = o \tag{14}$$

It is known that for the symmetric matrix $[e_{ij}]$, all the three roots are real. Putting $n = 1, 2, 3,$ in *(13)* and eliminating A_1, A_2, A_3 we get

$$\begin{vmatrix} I^{(n)} & x_1^{\,n} & x_2^{\,n} & x_3^{\,n} \\ I^{(1)} & x_1 & x_2 & x_3 \\ I^{(2)} & x_1^2 & x_2^2 & x_3^2 \\ I^{(3)} & x_1^3 & x_2^3 & x_3^3 \end{vmatrix} = 0 \tag{15}$$

Using (5), (6), (9), (14), and (15), $I^{(n)}$ can be expressed in terms of I_1, I_2, I_3.

PARTICULAR CASE OF INCOMPRESSIBLE FLUIDS

For incompressible fluids $I_1 = 0$ so that

$$I^{(1)} = 0,\ I^{(2)} = -2I_2,\ I^{(3)} = 3I_3,\ I^{(4)} = 2I_2^2,\ I^{(5)} = -5I_2\, I_3,$$
$$I^{(6)} = 3I_3^2 - 2I_2^3,\ I^{(7)} = 7I_2^2\, I_3,\ I^{(8)} = -8I_2\, I_3^2 + 2I_2^4, \ldots\ldots\ldots \tag{16}$$

Since $I^{(2)}$ is essentially positive, I_2 is essentially negative. I_3 can be positive or negative. We note from (16) that $I^{(2)}$, I , I , ... are essentially positive while I , I , I , ... have the same sign as I_3. We now prove that $I^{(n)}$ is essentially positive when n is even and $I^{(n)}$ has the same sign as I_3 when n is odd and not equal to unity. This is easily seen from equation (12) which gives for incompressible fluids

$$I^{(n)} = - I_2 \, I^{(n-2)} + I_3 \, I^{(n-3)} \tag{17}$$

Let us assume that our theorem is true upto $I^{(n-1)}$. If n is even, $(n-2)$ is even and $(n-3)$ is odd, so that $I^{(n-2)}$ is positive and $I^{(n-3)}$ has same sign as I_3 so that from (17) we get that $I^{(n)}$ is positive. If n is odd, $I^{(n-2)}$ has the same sign as I_3 and $I^{(n-3)}$ is positive so that $I^{(n)}$ has the same sign as I_3. Then the result follows from induction.

From (4), we see that in the expression for ϕ, the coefficients of $\mu_1, \mu_3, \mu_5, \ldots$ are all positive, whereas the coefficients of $\mu_2, \mu_4, \mu_6, \ldots\ldots$ have the same sign as I_3.

We assume the hypothesis that the dissipation function should be positive for all deformation rates. For some of these I_3 will be negative and for others it will be positive. Thus we draw the conclusion that (1) will be a possible rheological equation for a real Reiner-Rivlin fluid if

$$\mu_1 , \mu_3 , \ldots\ldots\ldots , \mu_{2r+1} \geqslant 0$$

and

$$\mu_2 , \mu_4 , \ldots\ldots\ldots , \mu_{2r} = 0 \tag{18}$$

i.e. the expression (1) should contain odd coefficients of viscosity only and these should be non-negative. In particular, Reiner-Rivlin fluids of the type

$$T = - pI + \mu_1 D + \mu_2 D^2 \tag{19}$$

with constant values of μ_1 and μ_2 do not exist unless $\mu_2 = 0$.

Thus all problems using the equation (19) with $\mu_2 \neq 0$ have dealt with fluids which do not satisfy the above hypothesis. However, we can also say that only those deformation rates are possible for these fluids for which $\phi \geqslant 0$ or that this constitutive equation may not hold for all deformation rates. In fact experimentally negative values of μ_3 have been observed and we also have got cases where different constitutive equations are required for explaining flow behaviour for small and large shear rates. This stresses the need for more experimental investigations.

CASE OF INCOMPRESSIBLE FLUIDS WHEN PHENOMENOLOGICAL COEFFICIENTS ACCEPT POWER SERIES EXPANSIONS

Let us consider Reiner-Rivlin fluids having rheological equation

$$T = \mu_0 I + \mu_1 D + \mu_2 D^2, \tag{20}$$

where μ_0, μ_1, μ_2 are functions of the invariants of the form

$$\mu_k = \sum_{rst} A_{krst} \, I_1^{\,r} \, I_2^{\,s} \, I_3^{\,t} \tag{21}$$

For incompressible fluids r must be zero. Let $A_{k0st} = B_{kst}$

so that

$$\mu_k = \sum B_{kst} \, I_2^{\,s} \, I_3^{\,t} \left(k = 0, 1, 2\right) \tag{22}$$

From (4), *(16)* and (22), we get

$$\phi = -2 \sum_{s\,t} B_{1st}\, I_2^{s+1} \quad I_3^{t} + 3 \sum_{s\,t} B_{2st}\, I_2^{s}\, I_3^{t+1} \tag{23}$$

Thus for incompressible fluids of type (20), *(21)* and (22), we have

(i) there are no restrictions on B_{0st}. These can be positive or negative,

(ii)	$B_{1st} = 0$	for odd values of t,
(iii)	$B_{2st} = 0$	for even values of *t*,
(iv)	$B_{1st} > 0$	if *s* is even, *t* is even,
(v)	$B_{1st} < 0$	if s is odd, *t* is even,
(vi)	$B_{2st} > 0$	if s is even *t* is odd,
and *(vii)*	$B_{2st} < 0$	if *s* is odd *t* is even.

These give a complete set of restrictions on the coefficients for incompressible fluids characterised by *(20)*.

A method for finding restrictions on the material constants is illustrated below by considering a tenth order fluid. This procedure can be extended to any order incompressible fluid.

The constitutive equation for a tenth order incompressible fluid is given by

$$\begin{aligned} \tau_{ij} = {} & (-p + \lambda_1 I_2 + \lambda_2 I_2^2 + \lambda_3 I_2^3 + \lambda_4 I_2^4 + \lambda_5 I_2^5 + \lambda_6 I_3 + \lambda_7 I_3^2 \\ & + \lambda_8 I_3^3 + \lambda_9 I_2 I_3 + \lambda_{10} I_2 I_3^2 + \lambda_{11} I_2^3 I_3 + \lambda_{12} I_2^2 I_3^2 + \lambda_{13} I_2^3 I_3)\delta_{ij} \\ & + (\mu_1 I_2 + \mu_2 I_2^2 + \mu_3 I_2^3 + \mu_4 I_2^4 + \mu_5 I_3 + \mu_6 I_3^2 + \mu_7 I_3^3 + \mu_8 I_2 I_3 \\ & + \mu_9 I_2 I_3^2 + \mu_{10} I_2^2 I_3 + \mu_{11} I_2^3 I_3)\, e_{ij} \\ & + (\nu_1 I_2 + \nu_2 I_2^2 + \nu_3 I_2^3 + \nu_4 I_2^4 + \nu_5 I_3 + \nu_6 I_3^2 \\ & + \nu_7 I_2 I_3 + \nu_8 I_2 I_3^2 + \nu_9 I_2^2 I_3)\, e_{ik}\, e_{kj} \end{aligned} \tag{24}$$

so the dissipation function is

$$\begin{aligned} \phi = {} & (\mu_1 I_2 + \mu_2 I_2^2 + \mu_3 I_2^3 + \mu_4 I_2^4 + \mu_5 I_3 + \mu_6 I_3^2 + \mu_7 I_3^3 \\ & + \mu_8 I_2 I_3 + \mu_9 I_2 I_3^2 + \mu_{10} I_2^2 I_3 + \mu_{11} I_2^3 I_3)(-2 I_2) \\ & + (\nu_1 I_2 + \nu_2 I_2^2 + \nu_3 I_2^3 + \nu_4 I_2^4 + \nu_5 I_3 + \nu_6 I_3^2 \\ & \quad + \nu_7 I_2 I_3 + \nu_8 I_2 I_3^2 + \nu_9 I_2^2 I_3)\, 3 I_3 \end{aligned} \tag{25}$$

$$\begin{aligned} = {} & -2\mu_1 I_2^2 - 2\mu_2 I_2^3 - 2\mu_3 I_2^4 - 2\mu_4 I_2^5 + (-2\mu_5 + 3\nu_1) I_2 I_3 \\ & + (-2\mu_6 + 3\nu_7) I_2 I_3^2 + (-2\mu_7 + 3\nu_8) I_2 I_3^3 + (-2\mu_8 + 3\nu_2) I_2^2 I_3 \\ & + (-2\mu_9 + 3\nu_9) I_2^2 I_3^2 + (-2\mu_{10} + 3\nu_3) I_2^3 I_3 + (-2\mu_{11} + 3\nu_4) I_2^4 I_3 \end{aligned} \tag{26}$$

Now I_2 is negative and I_3 can be positive or negative. Thus if ϕ is to be non-negative, we get the condition

$$\begin{aligned} & \mu_1 < 0,\ \mu_2 > 0,\ \mu_3 < 0,\ \mu_4 > 0,\ 2\mu_5 = 3\nu_1, \\ & 2\mu_6 - 3\nu_7 > 0,\ 2\mu_7 = 3\nu_8,\ 2\mu_8 = 3\nu_3, \\ & 2\mu_9 - 3\nu_9 < 0,\ 2\mu_{10} = 3\nu_3 \text{ and } 2\mu_{11} = 3\nu_4 \end{aligned} \tag{27}$$

(27) gives the restrictions on the thirty three material constants which are involved in the constitutive equation *(24)*.

REFERENCE

1. ERINGEN, A.C., "Non-linear Theory of Continuous Media" (McGraw Hill Book Co., Inc., New York) 1962, p. 170.