A NOTE ON THE OPTIMISATION OF A DEFENCE SYSTEM AGAINST AN AIR ATTACK

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The problem of determining the optimum mix of the defence weapons and the deployment of the mix has been dealt by Leibowitz & Lieberman. But the assumptions therein make it of less practical utility. In this note, an attempt has been made for a more realistic model. The mathematical expression for the "Measure of Effectiveness" has been modified and this, in turn, brings about a change in the pay-off function and the value of the game. A numerical example has also been worked out to illustrate the same.

Leibowitz & Lieberman[1] have defined the "Measure of Effectiveness"* as follows:

With a given level of effort, the attacker will try to maximise the damage to the target. This damage is a monotonically increasing function of the number of bombs surviving the defence. Similarly the defence, within the limitations of his budget, will try to minimise the number of surviving bombload. Therefore, the number of bombs surving the defence is taken as the measure of effectiveness for both the attack and the defence.

The mathematical expression for the "Measure of Effectiveness" has been modified by assigning a certain probability p_k to the defence weapons system. It gives the reliability of the system. This has been further improved by making allowance for the failure of each attack vehicle in view of the fact that the vehicles deteriorate with time[2]. To this end, a small probability has been attached with each vehicle for its failure.

The following symbols have been used:

$M_\circ$ = The level of effort defined in terms of the attacker's operational budget.

m = the number of the modes of attack.

j = The index of jth mode of attack, where $j = 1, 2, \ldots\ldots\ldots m$.

e_j = the destructive potential per attack vehicle in the jth mode of attack.

c_j = the cost per attck vehicle in the jth mode of attack.

$M_j = \dfrac{M_\circ}{c_j}$ = the number of vehicles in the jth mode of attack.

K_{ij} = the expected number of kills which can be achieved by a single defence weapon of ith type when it is completely reliable in the jth type of attack mode.

n = the number of different types of defence weapons and each being characterised by an index i, where $i = 1, 2, \ldots\ldots\ldots n$.

N_i = the number of units of the ith type defence weapon.

R_i = the radius of the circle with its centre at the target on which all the units of the ith type are deployed at equal distances.

*It is recommended that the reader should first go through the paper under reference 1.

$K_{ij}(R_i)$ = the effeciveness of a single ith type defence unit, deployed at a radius R_i against a jth type raid and expressed in terms of expected number of kills achieved by the defence unit before the raid reaches the bomb release line.

p_k = the reliability of the kth mix of defence weapons.

$p_x(j)$ = the probability of failure of the xth vehicle in the jth mode of attack.

MODIFIED BALANCED MATHEMATICAL MODEL AND ITS SOLUTION

The problem is to find the composition and deployment which will minimise the expected number of surviving bombs. Consider a situation in which the enemy elects to employ a j-type attack. The defence, is then faced with M_j vehicles each carrying e_j bombs. If the defender chooses the kth ($k=1,2,\ldots\ldots\ldots s$) feasible mix then the defence consists of $N_1^{(k)}$ units of the first type of weapon, $N_i^{(k)}$ units of the ith type. Although the defence weapons may be deployed in an inifinite number of ways, yet for each of the expositions the number of deployment is treated as finite. Thus the yth deployment indicates that all the first type weapons are deployed at radius $R_1^{(y)}$ all the second type of weapons at $R_2^{(y)}$ and in general all the ith type weapons at $R_i^{(y)}$ ($y = 1,2\ldots\ldots\ldots r$). If the defence has chosen the kth mix and the yth deployment, the expected number of kills achieved by the entire defence will be

$$Q_{kyj} = p_k \sum_{i=1}^{n} N_i^{(k)} K_{ij}\left(R_i^{(y)}\right) \tag{1}$$

where p_k is the reliability of the defence system. Leibowitz & Leiberman, in their analysis, have ignored this factor.

Again the expected number of the attacker's vehicles failing due to the mechanical defects is given by

$$\sum_{x=1}^{M_j} p_x^{(j)} \tag{2}$$

It has been assumed that there is always a certain chance for every vehicle to go out of order.

Thus the expected number of attack vehicles surviving the defence is

$$M_j - Q_{kyj} - \sum_{i=1}^{M_j} p_x^{(j)} = \frac{M_o}{c_j} - p_k \sum_{i=1}^{n} N_i^{(k)} K_{ij}\left(R_i^{(y)}\right) - \sum_{x=1}^{M_j} p_x^{(j)} \quad (3$$

However, if we assume that all the $p_x^{(j)}$ are equal, (say equal to p_j), (3) reduces to

$$\frac{M_\circ}{c_j} - p_k \sum_{i=1}^{n} N_i^{(k)} \ K_{ij}(R_i^{(y)}) - M_j p_j$$

$$= \frac{M_\circ}{c_j}(1 - p_j) - p_k \sum_{i=1}^{n} N_i^{(k)} \ K_{ij} \ (R_i^{(y)}) \qquad (4)$$

It can easily be seen from (4) that Leibowtz & Leiberman's expression is a particular case of (4) with $p_j = 0$ and $p_k = 1$

Finally the expected number of bombs penetrating the defence B_{kyj} is simply

$$B_{kyj} = e_j \left\{ \frac{M_\circ}{c_j}\left(1 - p_j\right) - Q_{kyj} \right\} \qquad (5)$$

Now for fixed k, the game B_k can be expressed as a matrix $\| B_{kyj} \|$ where

$$\left\| B_{kyj} \right\| = \left\| e_j \left\{ \left(1 - p_j\right) \frac{M_\circ}{c_j} - Q_{kyj} \right\} \right\| \qquad (6)$$

The value B_k may then be found out by treating the B_{kyj} matrix as a dual linear-programming problem[3]. Corresponding to each composition a value B_k will exist and the one with the least B_k would be considered as the best composition.

NUMERICAL EXAMPLE

Considering the example of Leibowitz & Leiberman, let us take

$$(p_k)_{k=1} = 0 \cdot 90$$
$$(p_k)_{k=2} = 0 \cdot 95$$
$$(p_k)_{k=3} = 0 \cdot 85$$

and $(p_j)_{j=1} = 0 \cdot 1$, $(p_j)_{j=2} = 0 \cdot 15$.

The changed elements of the matrices are as follows:

		$j=1$	$j=2$
$k = 1$	$y = 1$	12·6	3·1
	$y = 2$	10·8	6·7

		j 1	$j=2$
$k = 2$	$y = 1$	6·9	5·15
	$y = 2$	5·95	6·6

		$j=1$	$j=2$
$k = 3$	$y = 1$	2 4	6·8
	$y = 2$	2·4	6·8

With the help of these matrices , the values of game are as follows:

k	Opitmal defence strategy	Optimal attack strategy	Value of game
1	(,1)	(1,0)	10·80
2	($\frac{1}{3}$, $\frac{2}{3}$)	($\frac{2}{3}$, $\frac{1}{3}$)	6.22
3	(Any)	(0,1)	6·80

From the above, it is seen that the value of the game corresponding to the optimum mix ($k=2$), comes out to be 6·22 while the value obtained by Leibowitz & Leiberman was 7·33. Thus, with the help of the modified model, the damage to the target has been reduced by about 15 per cent.

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