RADIATIVE TRANSFER EFFECTS ON THE THERMAL INSTABIIITY OF A FLUID SPHERICAL SHELL

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(*Received* 18 *April* 1966)

The thermal instability of an incompressible fluid shperical shell heated within when the fluid absorbs and emits thermal radiation is considered. Two asymptotic cases of the radiative transfer equation (i) when the fluid is optically thin and (ii) when it is optically thick have been examined. It is observed that radiative transfer for the transparent medium has a stabilising effect on the fluid motion, whereas for the opaque medium the fluid behaves like the non-radiative case.

The problems on the thermal instability under a number of externally impressed conditions have recently been studied by many workers. In most of the problems the origin of the instability is a potential unstable arrangement of the fluid resulting from a prevailing adverse temperature gradient. The earliest theoretical investigation by Rayleigh in 1916 relates to the behaviour of a fluid enclosed between two parallel plates heated from below. This problem was later studied by several workers *viz.* Wasiutynski[1], Jefferys & Bland[2] and Chandrasekhar.[3] The formulation of the last two differed in the choice of the dependent variable. Jefferys and Bland[2] obtained the solution by taking temperature as the dependent variable while Chandrasekhar[3], by taking velocity as the dependent variable. Here the problem is solved by taking the radial velocity as the dependent variable. Goody[4] was first to study the thermal radiative transfer effects introducing these in the classical problem of Rayleigh or Pellew and Southwell. More recently Murgai & Khosla[5] have examined the effect of radiative transfer in case of an ionised medium in the presence of a vertical magnetic field which is the extension of earlier works of Goody[4] and Chandrasekhar[3]. It is of interest to examine the radiative transfer effects in problems of stability relating to other geometries because of their Astrophysical and Terrestrial significance. An incompressible fluid spherical shell generating heat from within is considered here. The problem has been solved for the two following approximations of the radiative transfer equation, (*i*) *appropriate to a transparent medium*—The treatment of the problem has been restricted to a case in which we have assumed temperature gradient β to be constant. It is not possible to establish the variational principle for the variable β, hence the above assumption is made. However, it will be appropriate to remark here that the effect of the variable nature of β has been analysed by Murgai & Khosla[4] in case of fluids confined between two parallel plates showingthat it does not alter appreciably the value of the critical Rayleigh number and (*ii*) *appropriate to an opaque medium*—The radiative transfer behaves like molecular conduction and its effect can be taken into account by modifying the thermal diffusivity. However in this case there is no need for any assumption in regard to the temperature gradient β.

BASIC EQUATIONS

Consider a homogeneous, incompressible fluid spherical shell of outer and inner radii R and R_1 respectively, under the effect of its own gravitation and with a uniform distribution of heat sources ϵ which maintains a radial temperature gradiant in the fluid. The fundamental equations of continuity, momentum, energy and radiative transfer are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \tag{1}$$

$$\rho \left[\frac{\partial \vec{U}}{\partial t} + (\vec{U} . \nabla) \vec{U} \right] = - \text{grad } P + \rho \nu \nabla^2 \vec{U} + \rho \nabla V \tag{2}$$

$$\frac{\partial T}{\partial t} + (\vec{U} . \nabla) T = K \nabla^2 T + \epsilon + \frac{\phi}{C_p} \tag{3}$$

$$\frac{dI}{ds} = k \left[B - I (s) \right] \tag{4}$$

$$\Phi = - \int \frac{dI}{ds} \, dw = - 4 \pi k B + k \int I (s) \, dw \tag{5}$$

$$\rho = \rho_o \left(1 - \alpha T \right) \tag{6}$$

$$\nabla V = - \frac{4}{3} \pi \rho G \vec{r} \tag{7}$$

where $\vec{U}$ is the velocity vector, ρ the density of the fluid, P the pressure, ν the kinematic viscosity, V the gravitational potential, T the temperature, K the thermal diffusivity, Cp the specific heat per unit volume, Φ the radiative heating per unit time, I the intensity of radiation at any point, k the absorption coefficient (inverse of mean free path of radiation), B the Plank function, S and ω are the elements of length and solid angle respectively, α the coefficient of volume expansion and G the gravitational constant. The temperature distribution is given by the energy equation in the static case.

$$K \left(\frac{d^2 T_o}{d r^2} + \frac{2}{r} \frac{d T_o}{d r} \right) + \epsilon + \Phi_o / C_p = 0 \tag{8}$$

where the quantities with subscript '0' refer to the equilibrium or static case.

We solve (8) by considering two asymptotic cases (*i*) when the fluid is optically thin and (*ii*) when it is optically thick. The two cases are characterised in terms of mean free path of radiation and are given by

$$\Phi = \begin{cases} - 4 \pi k B , & k R << 1 \quad (i) \\ \dfrac{4 \pi}{3 k} \nabla^2 B , & k R >> 1 \quad (ii) \end{cases} \tag{9}$$

The temperature of the outer surface has been assumed to be zero and so does not contribute to Φ in 9(*i*). Thus Φ_o may be written as

$$\Phi_0 = \begin{cases} - 4 \pi k \dfrac{\sigma T_o^4}{\pi} & k R << 1 \quad (i) \\ \dfrac{4 \pi}{3 k} \nabla^2 T_o^4 & k R >> 1 \quad (ii) \end{cases} \tag{10}$$

With the help of (10), (8) can be solved as

$$T_\circ = \begin{cases} \dfrac{\epsilon}{\lambda^2} \left\{ \dfrac{I_{\frac{1}{2}}\ (\lambda\, r)}{\sqrt{r}\, I_{\frac{1}{2}}\ (\lambda)} - 1 \right\} & k\, R << 1 \\ \beta_1\ (1 - r^2) & k\, R >> 1 \end{cases} \tag{11}$$

where

$$\beta_1 = \epsilon\ R^2/6\ K\ (1 + \chi)$$

$$\lambda^2 = \frac{4\ \pi\ k\ R^2\ S}{k\ C_p} = 3\ k^2\ R^2\ \chi\ ;\ \chi = \frac{4\ \pi\ S}{3\ k\ K\ C_p}\ ;\ S = \frac{\sigma\ T_\circ^{\ 3}}{\pi}$$

σ being Stefan's constant

In evaluating the temperature distribution, the two constants of integration have been determined from the condition that $T_\circ$ is zero at the surface and is finite at the centre. The equation (8) is linearized by assuming that the difference between the temperature at the centre and the surface is not large. This has been discussed in detail by Goody[4].

Now using Boussinesq approximation for the density variation, the linearised equations of the problem may be written as

$$\text{div}\ \vec{u} = 0 \tag{12}$$

$$\frac{\partial\ \vec{u}}{\partial\ t} = -\ \text{grad}\ (p/\rho_\circ) + \nu\ \nabla^2\ \vec{u} + \gamma\ \theta\ \vec{r} \tag{13}$$

$$\frac{\partial\ \theta}{t} = K\ \nabla^2\ \theta + 2\ \beta\ \vec{u}.\vec{r} + \phi/C_p \tag{14}$$

where ϕ, p, θ, u are the disturbances in temperature, pressure, radiative heating and velocity respectively. These have been assumed to be small

and
$$\gamma = \frac{4\ \pi}{3}\ \bar{\rho}\ G\ \alpha$$

Assuming a free surface both at $r = \eta < 1$ (a dimensionless radius) and $r = 1$, the boundary conditions to be satisfied by the solutions of equations (12), (13) and (14) will be

$$\left.\begin{aligned} \theta = U_r = 0 \\ \frac{d^2\ (r\ U_r\ )}{d\ r^2} = 0 \end{aligned}\right\} \quad \text{at } r = 1 \text{ and } r = \eta \tag{15}$$

MARGINAL STABILITY

Using the property of spherical symmetry we expand $\vec{U}.\vec{r}$ and θ in spherical harmonics

$$\begin{aligned} \vec{U}\ .\ \vec{r} &= W\ (r)\ Y_l^m\ (\Theta,\ \psi)\ e^{p^* t} \\ \theta &= \theta\ (r)\ Y_l^m\ (\Theta,\ \psi)\ e^{p^* t} \end{aligned} \tag{16}$$

where $\quad Y_l^m = P_l^m\ (\cos\ \Theta)\ e^{\pm\, i\, m\, \psi}$

and P^m are associated Legendre Polynomials.

Eliminating p from (13) by taking its curl and using (12) and (16) we have

$$D_l \left(D_l - \frac{p^* R^2}{\gamma} \right) W = \frac{\gamma}{\nu} l (l + 1) R^4 \theta \tag{17}$$

where

$$D_l = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{d r} - \frac{l (l + 1)}{r^2}$$

Also substituting for ϕ in terms of θ and using (16), the energy equation (14) in the two approximations can be written as :

$$(D_l - \lambda^2 - \sigma_1) \theta = \frac{-2\beta}{K} R^2 W , \quad k R << 1 \tag{18}$$

$$(D_l - \sigma_2) \theta = \frac{-2\beta}{K (1+\chi)} W , \quad k R >> 1 \tag{19}$$

where

$$\sigma_1 = \frac{p^* R^2}{K} \text{ and } \sigma_2 = \frac{p^* R^2}{K (1+\chi)}$$

The boundary conditions (15) can be expressed as

$$\left. \begin{aligned} W = \theta &= 0 \\ \frac{d^2W}{dr^2} &= 0 \end{aligned} \right\} \text{at } r = 1 \text{ and } r = \eta \tag{20}$$

The equations of marginal stability are characterised by $\frac{\partial}{\partial t} = 0$. Thus putting $p^* = 0$, in (17), (18) and (19) and eliminating θ from these, we get the equations of marginal stability

$$(D_l - \lambda^2) F = - l (l + 1) C_l W \quad k R << 1 \tag{21}$$

$$D_l F = \frac{- l (l + 1) C_l W}{(1 + \chi)} \quad k R >> 1 \tag{22}$$

$$D_l^2 W = F \tag{23}$$

where $C_l = \frac{2 \gamma \beta R^6}{\nu K}$ and is called the Rayleigh number. Its characteristic values determine the marginal state. β and γ have been assumed to be constants. For the opaque case β has been replaced by β_1 .

Equations (21) to (23) along with the boundary conditions (20) can now be solved by variational principle.

Multiplying (21) and (22) by r^2F and using (23), after integrating by parts, we get

$$l\ (l+1)\ C_l = \frac{\int_\eta^1 \left[r^2 \left(\frac{dF}{dr} \right)^2 + l\ (l+1)\ F^2 \right] dr + \lambda \int_\eta^1 r^2\ F^4\ dr}{\int_\eta^1 r^2 \left(D_l\ W \right)^2 dr} \qquad k\,R << 1 \tag{24}$$

and

$$l\,(l+1)\ C_l = \frac{\int_\eta^1 \left[r^2 \left(\frac{dF}{dr} \right)^2 + l\,(l+1)\ F^2 \right] dr \quad (1+\chi)}{\int_\eta^1 r^2 \left(D_l\ W \right)^2 dr} \qquad k\,R >> 1 \tag{25}$$

Now C_l has been expressed as the ratio of two positive definite integrals and it can be easily shown that corresponding to any arbitrary variation $\delta\, W$ in W compatible with the boundary conditions, the variation $\delta\, C_l$ in C_l is identically zero provided W satisfies the differential equations of the problem.

Since the form of (23) in this case agrees with the non-radiative stability problem studied by Chandrasekhar[3] we, therefore, can start with the same form of the function for F without any loss of generality for the radiative case as well. In this way an approximate value of $C_{l_{min}}$ can be found by minimising the integrals in (24) and (25) with respect to the parameters contained in the function for F. The accuracy of this minimum value can be increased by increasing the number of these parameters in the trial function. However, it is found that only one or two parameters are sufficient for a fairly accurate value of the Rayleigh number.

So, we put

$$F = \frac{1}{\sqrt{r}} \sum_j A_j\ \zeta_{l+\frac{1}{2},\,l+\frac{1}{2}}\,(\alpha_j\ r) \tag{26}$$

where

$$\zeta_{l+\frac{1}{2},\,l+\frac{1}{2}}(\alpha_j\ r) = J_{-(l+\frac{1}{2})}(\alpha_j\ \eta)\ J_{l+\frac{1}{2}}(\alpha_j\ r) - J_{l+\frac{1}{2}}(\alpha_j\ \eta)\ J_{-(l+\frac{1}{2})}(\alpha_j\ r) \tag{27}$$

$\zeta_{l+\frac{1}{2},\,l+\frac{1}{2}}$ denotes the cylinder function of order $l+\frac{1}{2}$, $J_{l+\frac{1}{2}}$ is the Bessel function of order $l+\frac{1}{2}$ and α_j is its jth zero. $A_j\ s$ are the various variational parameters. The value of the Rayleigh number can now be computed which for the first approximation comes out to be

$$l\ (l+1)\ C_l = \frac{(\alpha_1^2+\lambda^2)\,\alpha_1^6 \left\{ \frac{J^2_{l+\frac{1}{2}}(\alpha_1\,\eta)}{J^2_{l+\frac{1}{2}}(\alpha_1)} - 1 \right\}}{\alpha_1^2 \left\{ \frac{J^2_{l+\frac{1}{2}}(\alpha_1\ \eta)}{J^2_{l+\frac{1}{2}}(\alpha_1)} - 1 \right\} + \frac{4}{2l+1} \left[\frac{2l+3}{1-\eta^{2l+3}} \left\{ \frac{J_{l+\frac{1}{2}}(\alpha_1\,\eta)}{J_{l+\frac{1}{2}}(\alpha_1)} - \eta^{l+\frac{1}{2}} \right\}^2 - \frac{2l-1}{\eta^{-2l+1}-1} \left\{ \frac{J_{l+\frac{1}{2}}(\alpha_1\,\eta)}{J_{l+\frac{1}{2}}(\alpha_1)} - \eta^{-(l+\frac{1}{2})} \right\}^2 \right]} \qquad \text{for } k\,R << 1 \tag{28}$$

and

$$l(l+1)C_l = \frac{\alpha_1^8 \left\{ \frac{J^2_{l+\frac{1}{2}}(\alpha_1 \eta)}{J^2_{l+\frac{1}{2}}(\alpha_1)} - 1 \right\} (1+\chi)}{\alpha_1^2 \left\{ \frac{J^2_{l+\frac{1}{2}}(\alpha_1 \eta)}{J^2_{l+\frac{1}{2}}(\alpha_1)} - 1 \right\} + \frac{4}{2l+1} \left[\frac{2l+3}{1-\eta^{2l+3}} \left\{ \frac{J_{l+\frac{1}{2}}(\alpha_1 \eta)}{J_{l+\frac{1}{2}}(\alpha_1)} - \eta^{l+\frac{1}{2}} \right\}^2 - \frac{2l-1}{\eta^{-2l+1}-1} \left\{ \frac{J_{l+\frac{1}{2}}(\alpha_1 \eta)}{J_{l+\frac{1}{2}}(\alpha_1)} - \eta^{-(l+\frac{1}{2})} \right\}^2 \right]} \quad \text{for } kR >> 1 \qquad (29)$$

It is evident that in the limit of χ or $\lambda \to 0$, equations (28) and (29) reduce to Chandersekhar's[3] result. Tables 1 and 2 give the values of C_l for $\lambda = 10$ and $\lambda = 10^2$ respectively for the transparent case. In both these tables various vlues of l and η have been considered. The values of C_l / C°_l (for $\lambda = 10$) have been shown in Table 3, where C°_l is the value of the Rayleigh number obtained by Chandrasekhar[3]. Fig. 1 and 2 are the plot of log C_l *vs.* l for $\lambda = 10$ and 10^2 respectively. The various values of η have been shown on the curves. These calculations were performed for the transparent case only. In the opaque case the values of the Rayleigh number can easily be obtained from those given by Chandrasekhar[3] by multiplying them with $(1 + \chi)$.

This variational principle can similarly be established even when both the bounding surfaces are rigid or when one is rigid and the other is free.

CONCLUSION

The numerical results show that in case of transparent medium, for different values of λ in increasing order, the instability sets in at higher harmonics. Thus it can be concluded

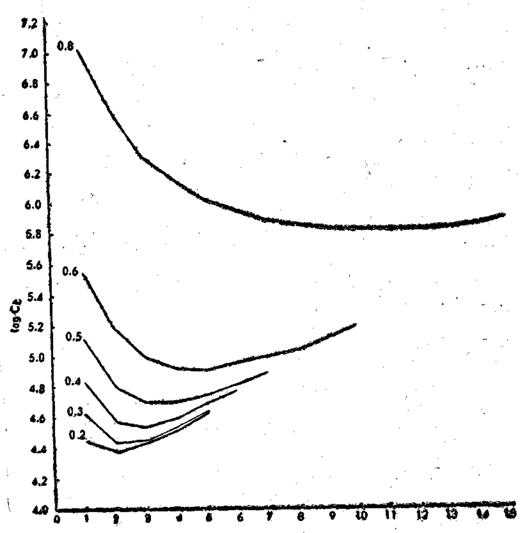

Fig. 1—Plot of log C_l *vs* l for $\lambda = 10$. Various curves are for values of η

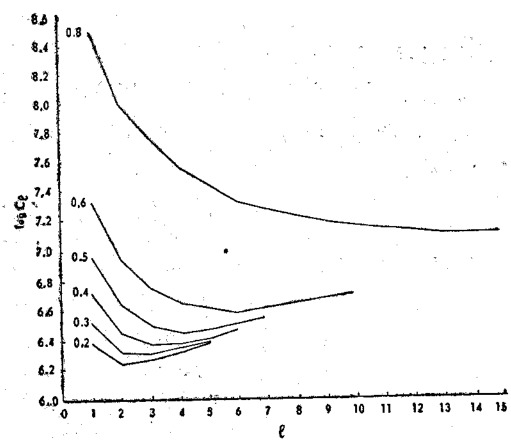

Fig. 2—Plot of log C_l *vs* l for $\lambda = 10^2$. Various curves are for values o

hat radiative transfer will try to damp out any disturbance of the fluid which may result due to heat transfer. In other words radiative transfer has got stabilising influence on the fluid. With the increase of λ from 10 to 10^2 this influence is also increased (Tables 1 & 2). This is also apparent from the ratio $C \;/ C^{\circ}_l$ where C°_l is the value of Rayleigh number in the absence of radiative transfer. The fact that the ratio decreases with l, the order of harmonic, shows that the effect of radiative transfer reduces for disturbances of higher harmonic modes (Table 3).

TABLE 1

THE VALUES OF C_l FOR DIFFERENT l AND η ; $\lambda = 10$

	η					
l	0·2	0·3	0·4	0·5	0·6	0·8
1	$2{\cdot}8966\times10^4$	$4{\cdot}1941\times10^4$	$6{\cdot}9463\times10^4$	$1{\cdot}3884\times10^5$	$3{\cdot}4498\times10^5$	$1{\cdot}0917\times10^7$
2	$*2{\cdot}2683\times10^4$	$*2{\cdot}7129\times10^4$	$3{\cdot}7893\times10^4$	$6{\cdot}4932\times10^4$	$1{\cdot}4793\times10^5$	$4{\cdot}1273\times10^6$
3	$2{\cdot}7042\times10^4$	$2{\cdot}8762\times10^4$	$*3{\cdot}4516\times10^4$	$5{\cdot}0492\times10^4$	$9{\cdot}9642\times10^4$	$2{\cdot}0734\times10^6$
4	$3{\cdot}4918\times10^4$	$3{\cdot}5524\times10^4$	$3{\cdot}8753\times10^4$	$*4{\cdot}9700\times10^4$	$8{\cdot}4404\times10^4$	$1{\cdot}3754\times10^6$
5	$4{\cdot}5444\times10^4$	$4{\cdot}5704\times10^4$	$4{\cdot}7387\times10^4$	$5{\cdot}5684\times10^4$	$*8{\cdot}2621\times10^4$	$*1{\cdot}0352\times10^6$
6			$5{\cdot}9719\times10^4$	$6{\cdot}5374\times10^4$	$8{\cdot}8071\times10^4$	$8{\cdot}4976\times10^5$
7				$7{\cdot}9689\times10^4$	$9{\cdot}8918\times10^4$	$7{\cdot}4393\times10^5$
8					$1{\cdot}1467\times10^5$	$6{\cdot}8398\times10^5$
9					$1{\cdot}3536\times10^5$	$6{\cdot}5339\times10^5$
10					$1{\cdot}6129\times10^5$	$6{\cdot}4372\times10^5$
11						$6{\cdot}4957\times10^5$
12						$6{\cdot}6845\times10^5$
13						$6{\cdot}9852\times10^5$
14						$7{\cdot}3954\times10^5$
15						$7{\cdot}9056\times10^5$

*These are critical Rayleigh Numbers.

TABLE 2

THE VALUES OF C_l FOR DIFFERENT l AND η ; $= \lambda = 10^2$

	η					
l	0·2	0·3	0·4	0·5	0·6	0·8
1	$2{\cdot}3802\times10^6$	$3{\cdot}3523\times10^6$	$5{\cdot}2871\times10^6$	$9{\cdot}7383\times10^6$	$2{\cdot}1058\times10^7$	$3{\cdot}2041\times10^8$
2	$*1{\cdot}7026\times10^6$	$2{\cdot}0087\times10^6$	$2{\cdot}7095\times10^6$	$4{\cdot}3341\times10^6$	$8{\cdot}6984\times10^6$	$1{\cdot}1949\times10^8$
3	$1{\cdot}8249\times10^6$	$*1{\cdot}9306\times10^6$	$*2{\cdot}2678\times10^6$	$3{\cdot}1448\times10^6$	$5{\cdot}5238\times10^6$	$5{\cdot}8836\times10^7$
4	$2{\cdot}1054\times10^6$	$2{\cdot}1382\times10^6$	$2{\cdot}3060\times10^6$	$*2{\cdot}8456\times10^6$	$4{\cdot}4018\times10^6$	$3{\cdot}8024\times10^7$
5	$2{\cdot}4445\times10^6$	$2{\cdot}4573\times10^6$	$2{\cdot}5339\times10^6$	$2{\cdot}9011\times10^6$	$3{\cdot}9900\times10^6$	$2{\cdot}7727\times10^7$
6			$2{\cdot}8609\times10^6$	$3{\cdot}0805\times10^6$	$*3{\cdot}9094\times10^6$	$2{\cdot}1943\times10^7$
7				$3{\cdot}3868\times10^6$	$4{\cdot}0164\times10^6$	$1{\cdot}8441\times10^7$
8					$4{\cdot}2472\times10^6$	$1{\cdot}6215\times10^7$
9					$4{\cdot}5687\times10^6$	$1{\cdot}4768\times10^7$
10					$4{\cdot}9618\times10^6$	$1{\cdot}3837\times10^7$
11						$1{\cdot}3252\times10^7$
12						$1{\cdot}2925\times10^7$
13						$*1{\cdot}2787\times10^7$
14						$1{\cdot}2808\times10^8$
15						$1{\cdot}2949\times10^7$

*These are critical Rayleigh Numbers.

TABLE 3

THE RATIO C_l / C°_l FOR DIFFERENT l AND η ; $\lambda = 10$

	η					
l	0·2	0·3	0·4	0·5	0·6	0·8
1	5·5586	4·9325	4·1298	3·3152	2·4589	1·4016
2	3·9739	3·8140	3·4732	2·9772	2·4120	1·4992
3	3·0445	3·0111	2·8859	2·6243	2·2523	1·3823
4	2·4941	2·4877	2·4450	2·3159	2·0707	1·3685
5	2·1426	2·1447	2·1278	2·0832	1·9156	1·3521
6			1·9001	1·8721	1·7810	1·3344
7				1·7215	1·6672	1·3164
8					1·5725	1·2979
9					1·4945	1·2789
10					1·4299	1·2612
11						1·2437
12						1·2270
13						1·2112
14						1·1970
15						1·1838

As far as the optically thick medium is concerned it is found that for all values of χ, the instability sets-in at the same modes at it happens for non-radiative case.

It may also be mentioned that the earlier conclusion drawn by Chandrasekhar[3] is that the pattern of convection which manifests itself at marginal stability shifts progressively to harmonics of higher order as the thickness of the mantlle decreases is also true when the effects of radiative transfer are taken into account.

It is observed that in case of infinitely thin and infinitely thick optical media the problem of instability reduces to that of non-radiative case considered by Chandrasekhar.

ACKNOWLEDGEMENTS

We are grateful to Dr. R.R. Aggarwal, Assistant Directcr, Applied Mathematics Division and Dr. C.D. Ghildyal for their kind help and encouragement. We are also thankful to the Research and Development Organisation, Ministry of Defence, Government of India for permission to publish this paper.

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