MELTING OF THIN CYLINDRICAL TUBES

K. L. Ahuja and I. J. Kumar

Defence Science Laboratory, Delhi

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It has been shown that the use of a polynomial profile for the temperature distribution in the case of thin cylindrical tubes is justified in the Goodman's technique of heat balance integral. The above technique has been used to obtain an approximate solution to the problem of melting (solidification) in a tube which is subjected to a constant heat flux at the inner surface while its outer surface is kept insulated. The temperature history and the melting rate are studied for the time during which the melting proceeds. Lighthill's technique for rendering approximate solution uniformly valid has been used and the first three terms of the series solution have been expressed in terms of an intrinsically small parameter.

NOMENCLATURE

a = inner radius of the tube

b = outer radius of the tube

c = dimensionless length $= \dfrac{b-a}{a}$

c' = specific heat

d = ratio of the outer radius to the inner radius $= 1 + c$

k = thermal diffusivity $= K/\rho c'$

K = thermal conductivity

L = latent heat

Q = heat flux

r = radius of the tube

s = position of the melting front

t = time

t_δ = transit time

t_m = melting time

T = temperature

$-T_\circ$ = initial temperature; the melting temperature corresponds to $T = 0$

u_1 = dimensionless temperature $= \dfrac{K_1\, k_2\, T_1}{K_2\, k_1\, T_\circ}$

u_2 = dimensionless temperature $= \dfrac{T_2}{T_\circ}$

v, w = thermal energy defined in (50) and (51)

x = dimensionless radius $= \dfrac{r-a}{a\,c}$

δ = penetration depth

δ' = dimensionless penetration depth $= \dfrac{\delta}{ac}$

ϵ = dimensionless length representing the position of the melting front

τ = dimensionless time = $\dfrac{k_2\,(t - t_m)}{a^2\, c^2}$

ξ = dimensionless time = $\dfrac{k_2 t}{a^2\, c^2}$

α, μ, ν = dimensionless parameters defined respectively by $\dfrac{Q\,a\,c}{2\,K_2\,T_\circ}$, $\dfrac{K_2\,T_\circ}{\rho_2\,L\,k_2}$ and $\dfrac{k_1}{k_2}$

λ = dimensionless number = $\dfrac{8c}{5}$

θ = dimensionless temperature = $\dfrac{K_2 T_2}{Qac}$

ϕ = thermal energy = $\displaystyle\int_1^{1-\delta'} (1 + c\,x)\,\theta\,dx$

ρ = density

Subscripts

1 — quantities in the liquid region

2 — quantities in the solid region

0 — initial value.

The heat conduction problems accompanied by the change of phase are non-linear because they involve a moving boundary whose location is not known a priori. Closed analytical solutions for such problems are difficult to obtain. A few exact solutions mostly pertaining to problems in plane configuration have been dealt with Neumann & Stefan[1] for prescribed temperature conditions on the wall. For a given heat flux at the surface, Evans *et al*[2] have obtained a solution in the form of Taylor series.

Since problems involving change of phase are non-linear because of release or absorption of latent heat at the moving boundary, a few of them have also been worked out by numerical methods. Landau[3] has solved a one-dimensional plane melting problem by using a finite difference step-by-step method. Allen & Severn[4] have applied relaxation method to study the unidimensional solidification of a semi-infinite plane region and the inward solidification of circular cylinders.

Quite recently certain integral methods suggested by Biot[5,6] and Goodman[7] have been applied to both linear and non-linear problems. Biot's method known as 'variational method' has been applied to problems where non-linearity arises because of temperature dependent heat transfer properties. Biot & Dughaday[8] have used the technique in treating an ablation problem of half space subjected to a constant rate of heat input at the melting surface. Goodman has applied the heat balance integral method in solving a number of melting problems in half space. The method has also been used by Goodman and Shea[9] in solving the problem of melting of finite slabs.

This paper is an attempt to solve the problem of melting of cylindrical tubes by making use of the technique of heat balance integral. The study finds application in some heat transfer problems suggested in the manufacture of metallic tubes and refrigeration technology. In the problem considered here a prescribed heat flux is applied at the inner surface of the tube while the outer surface is kept insulated. It is required to study the propagation of the melting front and the temperature distribution in both the melted and unmelted regions. A quadratic temperature profile is used for the temperature distribution in each of the above regions. While dealing with problems with axial symmetry, Sparrow[10] has hinted that the use of the polynomial profile in integral methods leads to inaccurate results. This anomaly was later explained by Lardner & Pohle[11] who argued that solution obtained by the use of above profile does not tend to the proper form of steady state solution for large values of time and that the volume changes are not the same for equal increment Δr in the radius r of the cylinderical tube. However, these arguments do not preclude the applicability of the polynomial profile to the case of cylindrical tubes whose thickness is small in comparison to their radius and for which the steady state is far off (as in the case of melting with small heating rates).

The problem has been solved in three steps. In step 1, an axially symmetric problem in ordinary heat conduction is briefly discussed by the use of a polynomial profile and results compared with those available in literature. A good agreement is found between the two results thus justifying the use of polynomial profile in thin annular regions. In step 2 the temperature distribution in the tube at the time of melting has been obtained by the application of the heat balance integral method. This serves as the necessary initial condition for the melting problem. Step 3 deals with the statement of the melting problem and its solution. Finally, the results have been discussed and depicted/graphically.

STEP. 1. *Heat conduction problem*—The temperature distribution in an infinite hollow cylinder which is subjected to a constant heat flux at its outer surface while its inner surface is maintained at zero temperature is considered. The equations governing the heat flow in the cylinder together with the initial and the surface conditions in the nondimensional form are given as

$$\frac{\partial \theta}{\partial \xi} = \frac{1}{1+cx} \frac{\partial}{\partial x} \left\{ (1+cx) \frac{\partial \theta}{\partial x} \right\} \tag{1}$$

$$\frac{\partial \theta}{\partial x} = 1 ; \qquad x = 1 , \xi > 0 \tag{2}$$

$$\theta = 0 ; \qquad x = 0 , \xi > 0 \tag{3}$$

$$\theta = 0 ; \qquad \xi = 0 \tag{4}$$

Let use suppose that at any instant of time the thermal layer has grown to a thickness δ' so that for $x \leqslant 1 - \delta'$, $\theta = 0$ and $\dfrac{\partial \theta}{\partial x} = 0$. Integrating (1) w.r.t. x between $x = 1$ and $x = 1 - \delta'$ the heat balance integral can be written as

$$(1 + c - c\delta') \frac{\partial}{\partial x} \theta(1 - \delta',\xi) - (1 + c) \frac{\partial}{\partial x} \theta(1,\xi) = \frac{d\phi}{d\xi} + (1+c-c\delta')\theta(1-\delta',\xi) \frac{d\delta'}{d\xi} \tag{5}$$

Using the boundary conditions at $x=1-\delta'$ and $x=1$ and integrating the resulting equation w.r.t. ξ, we get

$$\phi = -(1 + c)\, \xi + \text{constant.} \tag{6}$$

Let us assume that θ has the form

$$\theta = \frac{(1 - \delta' - x)^2}{2\delta'} \tag{7}$$

Thus θ satisfies the boundary conditions at $x=1-\delta'$ and $x=1$ and the corresponding ϕ is given by

$$\phi = -\frac{\delta'^2}{24}\left\{4\,(1 + c) - c\delta'\right\} \tag{8}$$

As the thermal layer reaches the inner boundary $x=0$ the boundary condition at (3) has to be satisfied. The temperature distribution and the transit time at that instant are given by

$$\theta = \frac{x^2}{2} \tag{9}$$

$$t_\delta = \frac{4 + 3c}{24\,(1 + c)} \tag{10}$$

When further heating is continued the temperature of the body rises and we have to find out the temperature distribution from (1) subject to the boundary conditions (2) and (3) and the initial condition (9), the origin of time being $\xi = t_\delta$. The heat balance integral approximation is introduced again with a quadratic temperature profile. Thus the temperature distribution in the cylinder is obtained as

$$\theta = x+(x^2-2x)\left[\frac{1+c}{2}\exp\left\{-\frac{24}{8+5c}(\xi-t_\delta)\right\}-\frac{c}{2}\right];\ t_\delta<\xi \tag{11}$$

From (11) the temperature at the outer surface of the cylinder is given by

$$\frac{K_2T_2}{aQ}=\frac{d'^2-1}{2}-\frac{d'(d'-1)}{2}\exp\left\{-\frac{24}{(3+5d')(d'-1)^2}\left(\frac{k_2t}{a^2}-\frac{(1+3d')(d'-1)^2}{24d'}\right)\right\};$$

$$\frac{(1+3d')\,(d'-1)^2}{24d'}<\frac{k_2t}{a^2} \tag{12}$$

The temperature time history at the outer surface of the cylinder from the exact solution is given by[12]

$$\frac{K_2T_2}{aQ}=d'\log d'-\pi\sum_{n=1}^{\infty}\exp\left(-\frac{k_2\,\beta_n^{\,2}t}{a^2}\right)J_1(d'\beta_n)J_0(\beta_n)$$

$$\left[\frac{J_0(\beta_n)Y_0(d'\beta_n)-Y_0(\beta_n)\,J_0(d'\beta_n)}{\beta_n\left\{J_0^2(\beta_n)-J_1^2(d'\beta_n)\right\}}\right] \tag{13}$$

where β_n are the roots of

$$J_0(\beta)\,Y_1(d'\beta)-Y_0(\beta)\,J_1(d'\beta)=0$$

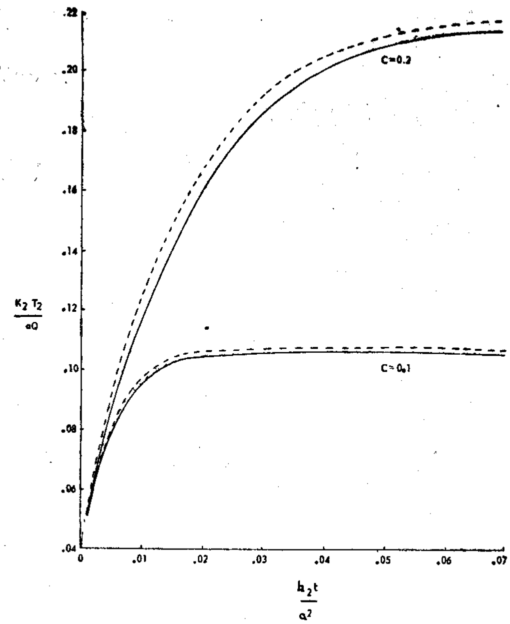

FIG. 1—Temperature time history at the outer surface of the hollow cylinder. Dotted curves represent the solution obtained by integral method and continuous curves represent the exact solution.

The results given at (12) and (13) have been evaluated and depicted graphically in Fig. 1 for $c = 0{\cdot}1$ and $c = 0{\cdot}2$

A good agreement has been found between the two results, difference being of the order of about 3% for $c=0{\cdot}1$ and about 6 % for $c=0{\cdot}2$. It is concluded therefore, that the use of a polynomial profile in thin annular regions is justified.

STEP 2: *Premelting temperature distribution*—Consider a hollow cylindrical tube of inner and outer radii a and b respectively held initially at a constant temperature $-T_\circ$. Let a constant heat input Q be then given to the inner surface. The flow of heat is governed by

$$\frac{\partial T_2}{\partial t} = \frac{k_2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T_2}{\partial r}\right); \; a < r < b, \; t > 0 \tag{14}$$

The boundary condition to be satisfied at the inner surface is*

$$K_2 \frac{\partial T_2}{\partial r} = -Q \;; \qquad r = a, \quad t > 0 \tag{15}$$

Let us now define a quantity $\delta(t)$ called the penetration depth, such that for $r \geqslant a+\delta$ the tube, for all practical purposes, is at the equilibrium temperture and there is no transfer of heat beyond this point. Hence at $r=a+\delta$, the following conditions must be satisfied

$$T_2 = -T_\circ \tag{16}$$

$$\frac{\partial T_2}{\partial r} = 0 \tag{17}$$

Initially at $t = 0, \delta(t) = 0$

The technique of the heat balance integral is now introduced to obtain the temperature distribution T_2 and penetration thickness δ, which satisfy the boundary conditions (15) to (17).

Integration of (14) w.r.t. r from $r = a$ to $r = a + \delta$ gives

$$\frac{k_2\, aQ}{K_2} = \frac{d\theta_2}{dt} + (a+\delta)\, T_0 \frac{d\delta}{dt} \tag{18}$$

where

$$\theta_2 = \int_a^{a+\delta} r\, T_2\, dr \tag{19}$$

*The problem of solidification can be treated similarly by changing the sign of Q.

To obtain the value of θ_2 we assume that T_2 can be expressed by a simple polynomial of the form

$$T_2 = A_1 + A_2 r + A_3 r^2 \tag{20}$$

where A_s may be functions of time. (20) with (15) to (17) gives

$$T_2 = - T_\circ + \frac{Q}{2K_2\delta} \left\{ \delta - (r - a) \right\}^2 \tag{21}$$

With the form of T_2 given by (21), (19) gives

$$\theta_2 = - \delta \left(a + \frac{\delta}{2} \right) T_0 + \frac{\delta^2(4a + \delta)Q}{24K_2} \tag{22}$$

(18) and (22) together give

$$\frac{k_2 a Q}{K_2} = \frac{\delta(8a + 3\delta)Q}{24 K_2} \frac{d\delta}{dt} \tag{23}$$

This differential equation together with its initial condition is integrated w.r.t. t directly to give

$$k_2 a t = \frac{\delta^2 (4a + \delta)}{24} \tag{24}$$

Till the thermal layer grows to the outer surface the temperature distribution is given by (21) and (24). However, when δ becomes equal to $(b-a)$ we have to specify different boundary condition at $r = b$. In this paper we consider that

$$\frac{\partial T_2}{\partial r} = 0 \quad ; \quad r = b \tag{25}$$

If t_m is the time when melting begins at $r=a$ and t_δ is the time when $\delta = (b-a)$, then t_m and t_δ from (21) and (24) are given as

$$k_2\, a\, t_m = \frac{1}{24} \left(\frac{2K_2 T_0}{Q} \right)^2 \left\{ \frac{2K_2 T_0}{Q} + 4a \right\} \tag{26}$$

$$k_2 a\, t_\delta = \frac{1}{24} (b - a)^2 \left\{ (b - a) + 4a \right\} \tag{27}$$

If $\dfrac{(b - a)Q}{2K_2 T_0} = \alpha$, then melting will not begin at the surface before the thermal layerr eaches the outer boundary $r = b$ provided $\alpha \leqslant 1$. In the problem discussed here we are interested in such heating which would satisfy this condition.

When $\delta=(b-a)$, the temperature distribution from (21) is given by

$$T_2 = - T_\circ + \frac{(b - a)Q}{2K_2} \left\{ 1 - \frac{r - a}{b - a} \right\}^2 ; \; a \leqslant r \leqslant b, t = t_\delta \tag{28}$$

After the thermal layer has reached the outer surface, the problem is to determine the temperature distribution which satisfies (14), the boundary conditions (15) and (25) and the initial condition (28). The temperature T_2 is again assumed to be quadratic in r. Proceeding on similar lines, the heat balance integral is derived and solved. The temperature distribution T_2 is given by

$$T_2 = - T_0 + \frac{(b-a)Q}{2K_2} \left\{ 1 - \frac{r - a}{b - a} \right\}^2 + \frac{2a\,Qk_2}{K_2(b^2-a^2)} (t-t_\delta); \; t_\delta \leqslant t \leqslant t_m \tag{29}$$

The time t_m when melting begins at $r=a$ is obtained by setting $T_2 = 0$ in (29) and we get

$$t_m = \frac{(b-a)^2}{k_2} \left\{ \frac{1}{6} + \frac{c}{24} + \frac{(c+2)(1-\alpha)}{4\alpha} \right\} \tag{30}$$

The temperature distribution at $t=t_m$ is given by

$$T_2 = -\frac{(b-a)Q}{2K_2} \left\{ 1 - \left(1 - \frac{r-a}{b-a} \right)^2 \right\} \tag{31}$$

STEP 3: *The melting problem and its solution*—As the melting temperature is reached at the surface $r=a$, the solid begins to melt and we obtain two distinct regions characterised by the melted and unmelted materials. These regions are denoted by subscript 1 and 2 respectively. The equations governing the flow of heat are given by (14) and its analog for the region 1, together with the following initial and boundary conditions:

$$T_1 = 0 \quad ; \quad t = t_m \quad , \quad s = 0 \tag{32}$$

$$T_2 = -\frac{acQ}{2K_2} \left\{ 1 - \left(1 - \frac{r-a}{b-a} \right)^2 \right\} ; t = t_m , s = 0 \tag{33}$$

$$T_1 = T_2 = 0 \quad ; r = a + s \quad , \quad t > t_m \tag{34}$$

$$K_1 \frac{\partial T_1}{\partial r} - K_2 \frac{\partial T_2}{\partial r} = -L\rho_2 \frac{ds}{dt} \quad ; r = a + s, t > t_m \tag{35}$$

$$K_2 \frac{\partial T_2}{\partial r} = 0 \quad ; \quad r = b \quad , t > t_m \tag{36}$$

$$K_1 \frac{\partial T_1}{\partial r} = -Q \quad ; \quad r = a, t > t_m \tag{37}$$

Before proceeding to solve (14) and its analog for the liquid region together with the boundary conditions (32) to (37), they are made dimensionless with the help of the following non-dimensional variables:

$$x = \frac{r-a}{ac}, \tau = \frac{k_2(t-t_m)}{a^2c^2}, u_2 = \frac{T_2}{T_\circ}, u_1 = \frac{K_1 k_2 T_1}{K_2 k_1 T_\circ}, \epsilon = \frac{s}{ac}$$

Introduction of these variables into the governing equations reduces them to the following form:

$$\nu \frac{\partial}{\partial x}\left\{(1+cx)\frac{\partial u_1}{\partial x}\right\} = (1+cx)\frac{\partial u_1}{\partial \tau} ; 0 < x < \epsilon, \tau > 0 \tag{38}$$

$$\frac{\partial}{\partial x}\left\{(1+cx)\frac{\partial u_2}{\partial x}\right\} = (1+cx)\frac{\partial u_2}{\partial \tau} \quad ; \epsilon < x < 1, \tau > 0 \tag{39}$$

$$u_1 = 0 \quad ; \tau = 0 \quad ; \epsilon = 0 \tag{40}$$

$$u_2 = -\alpha \left\{1 - (1-x)^2\tau\right\} \quad ; \tau = 0, \epsilon = 0 \tag{41}$$

$$u_1 = u_2 = 0 \quad ; x = \epsilon , \tau > 0 \tag{42}$$

$$\nu \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} = -\frac{1}{\mu} \frac{d\epsilon}{d\tau} \quad ; x = \epsilon , \tau > 0 \tag{43}$$

$$\frac{\partial u_2}{\partial x} = 0 \; ; \quad x = 1 \quad , \tau > 0 \tag{44}$$

$$\frac{\partial u_1}{\partial x} = - \frac{2\alpha}{\nu} \quad ; \quad x = 0 \quad ; \tau > 0 \tag{45}$$

(42) expresses the fact that the temperature at the melting front is equal to the melting temperature and (43) gives the heat balance across it.

As done earlier, the technique of heat balance integral is applied to (38) and (39). Quadratic profiles are assumed for the temperature distribution for both melted and unmelted regions. This leads us to the following differential equations in τ

$$\mu \frac{dw}{d\tau} + \mu \frac{dv}{d\tau} + (1 + c\epsilon) \frac{d\epsilon}{d\tau} = 2\mu\alpha \tag{46}$$

$$(8 + 3c\epsilon)\epsilon^2 \frac{dw}{d\tau} = 2\alpha(12 + 8c\epsilon + c^2\epsilon^2)\epsilon^2 - 24\nu (1 + c\epsilon)w \tag{47}$$

$$(1 - \epsilon)^2(8+5c + 3c\epsilon) \frac{dv}{d\tau} = - 24(1 + c\epsilon) v \tag{48}$$

The initial conditions to be satisfied are

$$\epsilon(\tau = 0) = 0 \; ; w (\tau = 0) = 0 \; ; v (\tau = 0) = -\frac{8 + 5c}{12} \alpha \tag{49}$$

The quantities v and w occuring in the above equations are given by the following relations

$$v = \int_{\varepsilon}^{1} (1 + cx) \, u_2 dx \tag{50}$$

$$w = \int_{0}^{\varepsilon} (1 + cx) \, u_1 \, dx \tag{51}$$

and represent expressions for the thermal energy for regions 2 and 1 respectively. These equations may be solved as under

Since (46), (47) and (48) are coupled non-linearly their solutions cannot be obtained directly. In such situations the solution is expressed in power series in assending powers of some parameter. Since α by definition lies between 0 and 1, the series solution is expressed in powers of α. We assume the following series for ϵ, w and v:

$$\epsilon\,(c,\alpha,\mu,\nu,\tau) = \overset{(0)}{\epsilon}\,(c,\mu,\nu,\tau) + \alpha \overset{(1)}{\epsilon}\,(c,\mu,\nu,\tau) + \alpha^2 \overset{(2)}{\epsilon}\,(c,\mu,\nu,\tau) + \ldots \tag{52}$$

$$w\,(c,\alpha,\mu,\nu,\tau) = \overset{(0)}{w}\,(c,\mu,\nu,\tau) + \alpha \overset{(1)}{w}\,(c,\mu,\nu,\tau) + \alpha^2 \overset{(2)}{w}\,(c,\mu,\nu,\tau) + \ldots \tag{53}$$

$$v\,(c,\alpha,\mu,\nu,\tau) = \overset{(0)}{v}\,(c,\mu,\nu,\tau) + \alpha \overset{(1)}{v}\,(c,\mu,\nu,\tau) + \alpha^2 \overset{(2)}{v}\,(c,\mu,\nu,\tau) + \ldots \tag{54}$$

Substituting for ϵ, w and v from (52), (53) and (54) in (46), (47) and (48) we may obtain solutions of different orders. The zeroth order solution is given by

$$\overset{(0)}{\epsilon} = 0 \; ; \; \overset{(0)}{w} = 0 \text{ and } \overset{(0)}{v} = 0 \tag{55}$$

and is in conformity with the physical assumption. The first order solution is likewise given by

$$\left.\begin{aligned} \overset{(1)}{\epsilon} &= -\mu \overset{(1)}{v} + 2\mu\left\{\tau - \frac{8+5c}{24}\right\} \\ \overset{(1)}{v} &= -\frac{8+5c}{12}\exp\left(-\frac{24\tau}{8+5c}\right) \\ \overset{(1)}{w} &= 0 \end{aligned}\right\} \tag{56}$$

This solution is uniformly valid and well behaved. However, when second order solutions are calculated terms of the form $\tau \exp\left(-\dfrac{24\,\tau}{8+5c}\right)$ and $\tau^2 \exp\left(-\dfrac{24\,\tau}{8+5c}\right)$ are obtained in $\overset{(2)}{v}$. The effect of these terms is to cause humps in the graph of $\overset{(2)}{v}$ *vs* τ or essentially v *vs* τ. The solution, therefore, does not behave properly and is thus not uniformly valid. To avoid this difficulty we resort to Lighthill's[13] technique for "rendering approximate solutions uniformly valid". The essence of this technique is to expand the independent variable in terms of another independent variable and in powers of the known small parameter. The terms in the expansion of the original independent variable are chosen in such a way that they eliminate the undesirable terms in the expansion of the dependent variables. This is achieved in the following way.

Let η be the new independent variable and let the original independent variable τ admit an expansion of the form

$$\tau(c,\alpha,\mu,\nu,\eta) = \eta + \alpha\,\overset{(1)}{\tau}(c,\mu,\nu,\eta) + \alpha^2\,\overset{(2)}{\tau}(c,\mu,\nu,\eta) + \ldots \tag{57}$$

In the expansion for ϵ, w and v, τ is replaced by η and the governing equations (46) to (48) in terms of the new independent variable are written as

$$\mu\frac{dw}{d\eta} + \mu\frac{dv}{d\eta} + (1+c\epsilon)\frac{d\epsilon}{d\eta} = 2\,\mu\,\alpha\frac{d\tau}{d\eta} \tag{58}$$

$$(8+3c\epsilon)\epsilon^2\frac{dw}{d\eta} = \left\{2\,\alpha\,(12+8c\epsilon+c^2\epsilon^2)\,\epsilon^2 - 24\nu\,(1+c\epsilon)w\right\}\frac{d\tau}{d\eta} \tag{59}$$

$$(8+5c+3c\epsilon)(1-\epsilon)^2\frac{dv}{d\eta} = -24\,(1+c\epsilon)\,v\,\frac{d\tau}{d\eta} \tag{60}$$

It is assumed that $\tau^i(\eta) = 0$ at $\eta = 0$ for $i \geqslant 1$. The initial conditions (49) are correspondingly written as

$$\epsilon(\eta=0) = 0,\ w\,(\eta=0) = 0,\ v\,(\eta=0) = -\frac{8+5c}{12}\,\alpha \tag{61}$$

Proceeding on the above lines the expansions for ϵ, w and v are substituted in (58) to (60) to obtain solutions of various orders.

The zeroth order solution is given by

$$\overset{(0)}{\epsilon} = 0,\quad \overset{(0)}{w} = 0,\quad \overset{(0)}{v} = 0 \tag{62}$$

The first order solution is similarly given as under

$$\overset{(1)}{\epsilon} = 2\mu\left\{ \eta - \frac{8+5c}{24} + \frac{8+5c}{24} \exp\left(- \frac{24\,\eta}{8+5c} \right)\right\} \tag{63}$$

$$\overset{(1)}{v} = - \frac{8+5c}{12} \exp\left(- \frac{24\eta}{8+5c}\right) \tag{64}$$

$$\overset{(1)}{w} = 0 \tag{65}$$

The equations for the second order approximation can be written as

$$\mu \frac{d\overset{(2)}{w}}{d\eta} + \mu \frac{d\overset{(2)}{v}}{d\eta} + \frac{d\overset{(2)}{\epsilon}}{d\eta} + c\overset{(1)}{\epsilon} \frac{d\overset{(1)}{\epsilon}}{d\eta} = 2\mu \frac{d\overset{(1)}{\tau}}{d\eta} \tag{66}$$

$$\overset{(2)}{w} = 0 \tag{67}$$

$$\frac{d\overset{(2)}{v}}{d\eta} + \frac{24}{8+5c}\overset{(2)}{v} = \frac{2 \exp\left(- \frac{24\eta}{8+5c}\right)}{8+5c} \left\{ 2\mu\,(16+15c+5c^2) \left(\eta - \frac{8+5c}{24}\right) + 2\,\mu \left(\frac{8+5c}{24}\right) \left(16 + 15c + 5c^2\right) \exp\left(- \frac{24\eta}{8+4c}\right) + (8+5c) \frac{d\tau^{(1)}}{d\eta} \right\} \tag{68}$$

In order that $\overset{(2)}{v}$ may be free of undesirable terms

$$(8+5c) \frac{d\overset{(1)}{\tau}}{d\eta} + 2\mu\ (16+15c+5c^2) \left(\eta - \frac{8+5c}{24}\right) = 0 \tag{69}$$

This equation, on integration, gives

$$\overset{(1)}{\tau} = - \frac{\mu\,(16+15c+5c^2)}{8+5c}\ \eta \left(\eta - \frac{8+5c}{12}\right) \tag{70}$$

From (66) and (68) the approximations of the second order are given by

$$\overset{(2)}{\epsilon} = 2\mu\overset{(1)}{\tau} - \mu\overset{(2)}{v} - \frac{c}{2}\left(\overset{(1)}{\epsilon}\right)^2 \tag{71}$$

$$\overset{(2)}{v} = \frac{\mu(8+5c)\,(16+15c+5c^2)}{144} \left\{ \exp\left(- \frac{24\eta}{8+5c}\right) - \exp\left(- \frac{48\eta}{8+5c}\right)\right\} \tag{72}$$

It is now apparent from (72) that $\overset{(2)}{v}$ no longer contains terms of the form $\eta \exp\left(- \frac{24\eta}{8+5c}\right)$ and $\eta^2 \exp\left(- \frac{24\eta}{8+5c}\right)$ and is, therefore, free of undesirable terms.

This solution is well behaved and uniformly valid. Thus the Lighthill technique has helped in removing the undesirable terms.

Exactly in the above manner, equations for the third order approximations are derived and solved. The choice of $\overset{(2)}{\tau}$ which will render $\overset{(3)}{v}$ uniformly valid is suggested as

$$\overset{(2)}{\tau} = \frac{\mu^2}{450(1+\lambda)}\left[\left\{ 2(1+\lambda)\,(N-8M^2)\right\}\left\{\frac{1+\lambda}{3} - \left(\eta+\frac{1+\lambda}{3}\right)\exp\left(-\frac{3\eta}{1+\lambda}\right)\right\} + (N-4M^2)\,(1+\lambda)\eta - 3N\eta^2 + \frac{3N}{1+\lambda}\eta^3\right] \tag{73}$$

The corresponding values of $\overset{(3)}{\epsilon}$, $\overset{(3)}{v}$, and $\overset{(3)}{w}$ are

$$\overset{(3)}{\epsilon} = 2\mu\overset{(2)}{\tau} - c\,\overset{(1)}{\epsilon}\,\overset{(2)}{\epsilon} - \mu\,\overset{(3)}{v} - \frac{\mu}{\nu}\left(\overset{(1)}{\epsilon}\right)^2 \tag{74}$$

$$\overset{(3)}{v} = \frac{\mu^2(1+\lambda)}{1350}\left\{3\,(N-12M^2)\exp\left(-\frac{3\eta}{1+\lambda}\right) - 4\,(N-16M^2)\exp\left(-\frac{6\eta}{1+\lambda}\right) + (N-28M^2)\exp\left(-\frac{9\eta}{1+\lambda}\right)\right\} \tag{75}$$

$$\overset{(3)}{w} = -\frac{1}{\nu}\left(\overset{(1)}{\epsilon}\right)^2 \tag{76}$$

where

$$\lambda = \frac{5c}{8}$$

$$M = 10 + 15\lambda + 8\lambda^2$$

$$N = 64M\lambda(1+\lambda) + 40\,(5-\lambda)(1+\lambda) + 8M^2 - 4M\,(10+7\lambda)$$

Proceeding in a similar manner the approximations of any higher order can be obtained.

RESULTS AND DISCUSSION

With the help of the heat balance integral and Lighthill's technique for rendering approximate solutions uniformly valid we have been able to express τ, ϵ, v and w in the form of series in ascending powers of α and as functions of the new independent variable η and c, μ and ν. The series for τ after substituting for $\tau^{(1)}$ and $\tau^{(2)}$ from (70) and (73) respectively can be written as

$$\tau = \eta + \alpha\,\mu\,\frac{\tau^{(1)}(\eta,\,c)}{\mu} + \alpha^2\,\mu^2\,\frac{\tau^{(2)}(\eta,\,c)}{\mu^2} + \ldots\ldots \tag{77}$$

This is clearly a function of η, $\alpha\mu$ and c. The dimensionless melt thickness ϵ can be found from (63), (71) and (74) as

$$\epsilon = \alpha\,\mu\,\frac{\epsilon^{(1)}(\eta,\,c)}{\mu} + \alpha^2\,\mu^2\,\frac{\epsilon^{(2)}(\eta,\,c)}{\mu^2} + \alpha^3\mu^3\,\frac{\epsilon^{(3)}(\eta,\,c)}{\mu^3} + \ldots \tag{78}$$

The expanssions for v and w can similarly be given by the following equations :

$$v = \alpha \left\{ v^{(1)}(\eta, c) + \alpha\mu \frac{v^{(2)}(\eta, c)}{\mu} + \alpha^2\mu^2 \frac{v^{(3)}(\eta, c)}{\mu^2} + \ldots \right\} \quad (79)$$

$$w = \alpha \left\{ \frac{\alpha^2\mu^2}{\nu} \quad \frac{[\epsilon^{(1)}(\eta, c)]^2}{\mu^2} \right\} \quad (80)$$

From (78), ϵ is a function of $\alpha\mu$, c and η. From (77) it is possible to calculate physically meaningful dimensionless time τ from the fictitious time η for various values of $\alpha\mu$ and c. It is also clear from the above equations that the parameter ν occurs in the third order approximation and that α and μ occur always as $\alpha\mu$. Fig. 2 is a plot of ϵ vs $\frac{k_2(t-t_m)}{a^2}$ for values of $c=0{\cdot}2$, $\nu=0{\cdot}6$ and $\alpha\mu=0{\cdot}15$, $0{\cdot}30$ and $0{\cdot}40$. It is clear from Fig. 2 that when the values of $\alpha\mu$ are increased less time will be required for melting. Fig. 3 shows the time required to melt a tube for two different values of c and for $\alpha\mu = 0{\cdot}15$ and $\nu = 0{\cdot}6$ and $1{\cdot}0$.

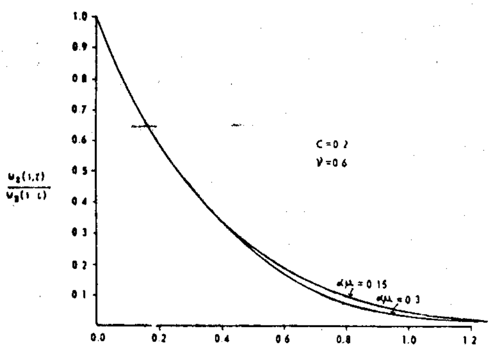

FIG. 2—Melt thickness vs time for $c=0{\cdot}2$, $\nu=0{\cdot}6$ and various values of $\alpha\mu$.

The temperature distribution in the melted and unlelted regions can now be determined from the non-dimensional form of (26) by applying boundary conditions (40) to (45) relations (50) and (51) and is given by

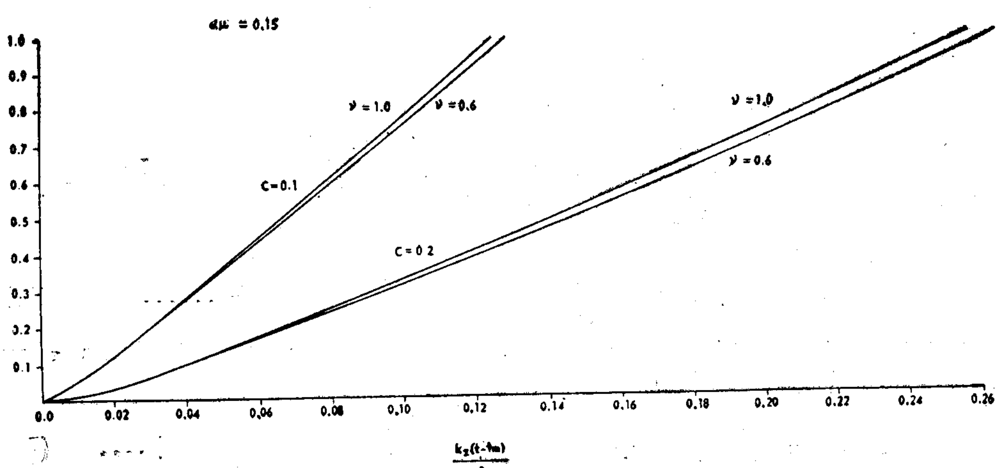

FIG. 3—Melt thickness vs time for $\alpha\mu = 0.15$, $c = 0.1$ and 0.2 and $\nu = 0.6$ and 1.0

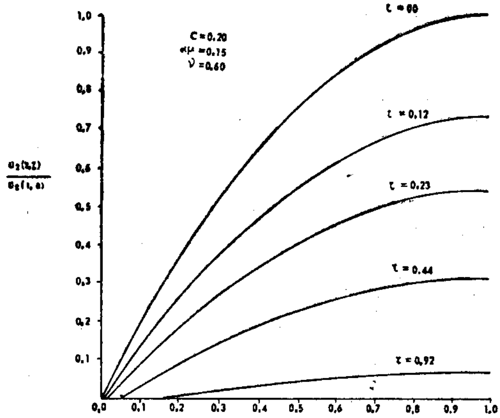

FIG. 4—Temperature distribution in the solid region for $c = 0{\cdot}2$, $\nu=0{\cdot}6$, $\alpha\mu = 0.15$ and various values of τ

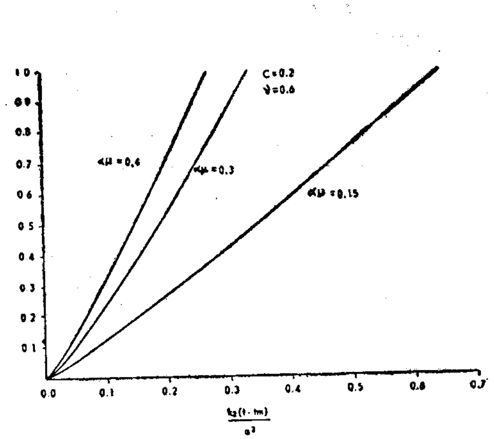

FIG. 5—Temperature time history at the outer surface of the cylinder for $c=0{\cdot}2$, $\nu=0{\cdot}6$ and $\alpha\mu = 0{\cdot}15$ and $0{\cdot}30$

$$u_1 = (\epsilon - x)\left[\frac{2\alpha}{\nu} - \frac{\epsilon + x}{(8 + 3\,c\,\epsilon)\epsilon^3}\left\{\frac{4\alpha}{\nu}(3 + c\,\epsilon)\,\epsilon^2 - 12\,w\right\}\right] \qquad o \leqslant x \leqslant \epsilon,\ \tau > o \quad (81)$$

$$u_2 = \frac{12\,\nu}{(8 + 5\,c + 3\,c\,\epsilon)\,(1 - \epsilon)}\left\{1 - \left(\frac{1 - x}{1 - \epsilon}\right)^2\right\};\ \epsilon \leqslant x \leqslant 1,\ \tau > o \quad (82)$$

With the help of the above equations it is possible to determine the temperature distribution in both melted and unmelted regions. Fig. 4 gives the temperature distribution $\dfrac{u_2(x, \tau)}{u_2(1, 0)}$ *vs* x for different values of τ and $c=0{\cdot}2$, $\alpha\mu = 0{\cdot}15$ and $\nu = 0{\cdot}6$. Fig. 5 gives the temperature time history at the insulated surface for values of $c = 0{\cdot}2$, $\nu = 0{\cdot}6$ and $\alpha\mu = 0{\cdot}15$ and $0{\cdot}30$. It is clear from Fig. 5 that the temperature of the solid portion of the tube is immediately raised to the melting temperature and the heat subsequently applied helps to produce melting and to raise the temperature of the molten portion.

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