THE INSTABILITY OF AN INFINITE HOMOGENEOUS ROTATING MEDIUM IN PRESENCE OF A MAGNETIC FIELD

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In the present paper the necessary and sufficient condition for the gravitational instability of a homogeneous rotating medium in presence of a magnetic field has been discussed. The medium is assumed to have finite electrical conductivity and viscosity. The criteria for stability comes out to be the Jeans criteria.

The conditions for the gravitational instability of homogeneous, non-viscous medium in presence of a magnetic field have been studied by Chandrasekhar and Fermi [1]. The problem was extended by Chandrasekhar[2] to include the effect of rotation. In both the cases it was found that the criteria for stability comes out to be the well known Jeans criteria. The case of a viscous medium has been studied by Stephenson[3]. In the present paper we have taken into consideration the finite electrical conductivity of the medium. The instability conditions for two particular cases have been discussed.

BASIC EQUATIONS

The basic equations of the system are:

$$\begin{aligned}
\rho\frac{du}{dt} &= \frac{1}{4\pi}\,\text{curl}\, h\times H+2\rho u\times\Omega-\text{grad}\,(\delta p)+\rho\,\text{grad}\,(\delta v)+\mu\nabla^2u\\
\frac{\partial h}{\partial t} &= \text{curl}\,(u\times H)+\frac{1}{4\pi\sigma}\nabla^2 h\\
\text{div}\, h &= 0 \qquad\qquad (1)\\
\frac{\partial}{\partial t}(\delta\rho) &= -\rho\,\text{div}\, u\\
\nabla^2(\delta v) &= -4\pi G\,\delta\rho
\end{aligned}$$

In these equations u is the velocity, ρ the density, H the magnetic field, Ω the angular velocity of the medium, μ the coefficient of viscosity, σ the electrical conductivity and δp, $\delta\rho$, δv and h are the perturbations in pressure, density, gravitational potential and magnetic field respectively.

We now suppose that the coordinate axes are so chosen that $H=(0, H_y, H_z)$ and $\Omega=(\Omega_x, \Omega_y, \Omega_z)$.

SOLUTION

The solutions of equations (1) will be sought corresponding to the propagation of waves in the Z—direction. div $h=0$ gives $h_z=0$. Equations (1) become

$$\frac{\partial h_x}{\partial t}-H_z\frac{\partial u_x}{\partial z}-\nu_m\frac{\partial^2 h_x}{\partial z^2}=0$$

$$\frac{\partial h_y}{\partial t}+H_y\frac{\partial u_z}{\partial z}-H\frac{\partial u_y}{\partial z}-\nu_m\frac{\partial^2 h_y}{\partial z^2}=0$$

$$\frac{\partial u_x}{\partial t}-\frac{H_z}{4\pi\rho}\frac{\partial h_x}{\partial z}+2(u\ \Omega_y-u_y\,\Omega_z)-\nu\frac{\partial^2 u_x}{\partial z^2}=0$$

$$\frac{\partial u_y}{\partial t} - \frac{H_z}{4\pi\rho}\ \frac{\partial h_y}{\partial z} + 2\left(u_x\,\Omega_z - u_z\,\Omega_x\right) - \nu\,\frac{\partial^2 u_y}{\partial z^2} = 0 \tag{2}$$

$$\frac{\partial u_z}{\partial t} + \frac{H_y}{4\pi\rho}\ \frac{\partial h_y}{\partial z} + 2\left(u_y\,\Omega_x - u_x\,\Omega_y\right) - \nu\,\frac{\partial^2 u_z}{\partial z^2} + \frac{c^2}{\rho}\cdot\frac{\partial}{\partial z}(\delta\rho) - \frac{\partial}{\partial z}(\delta v) = 0$$

$$\frac{\partial}{\partial t}(\delta\rho) + \rho\,\frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial^2}{\partial z^2}(\delta v) + 4\pi G\,(\delta\rho) = 0$$

Corresponding to a wave motion in Z-direction we now put $\frac{\partial}{\partial t} = i\omega$ and $\frac{\partial}{\partial z} = -ik$ where ω is the wave frequency and k is the wave number.

Equation (2) in Matrix form can be written as

$$\begin{vmatrix} (\omega - ik^2\nu_m) & 0 & kH_z & 0 & 0 & 0 & 0 \\ 0 & (\omega - ik^2\nu_m) & 0 & kH_z & -kH_y & 0 & 0 \\ \frac{kH_z}{4\pi\rho} & 0 & (\omega - ik^2\nu) & 2i\Omega_z & -2i\Omega_y & 0 & 0 \\ 0 & \frac{kh_z}{4\pi\rho} & 2i\Omega_z & (\omega - ik^2\nu) & 2i\Omega_x & 0 & 0 \\ 0 & \frac{-kH_y}{4\pi\rho} & 2i\Omega_y & -2i\Omega_x & (\omega - ik^2\nu) & \frac{-c^2k}{\rho} & 0 \\ 0 & 0 & 0 & 0 & -\rho k & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\pi G & -k^2 \end{vmatrix} \begin{vmatrix} h_x \\ h_y \\ u_x \\ u_y \\ u_z \\ \delta\rho \\ \delta v \end{vmatrix} = 0 \tag{3}$$

The condition that these equations possess a non-trivial solution is that the determinant of the (7×7) matrix should be zero. This condition gives

$$\begin{aligned} &\omega\,(\omega - ik^2\nu_m)^2(\omega - ik^2\nu)^3 - (\omega - ik^2\nu_m)^2(\omega - ik^2\nu)^2\,\Omega^2{}_J - 4\omega\,(\omega - ik^2\nu_m)^2(\omega - ik^2\nu)\Omega^2 \\ &+ 4\Omega^2{}_J\,\Omega^2{}_Z\,(\omega - ik^2\nu_m)^2 - \omega\,(\omega - ik^2\nu_m)\,(\omega - ik^2\nu)^2\,(2\Omega^2{}_A + \Omega^2{}_B) \\ &+ 2\,(\omega - ik^2\nu_m)(\omega - ik^2\nu)\,\Omega^2{}_A\,\Omega^2{}_J + 4\,\omega(\omega - ik^2\nu_m)\left[\Omega^2{}_A\,\Omega^2{}_X + (\Omega_A\,\Omega_Y - \Omega_B\,\Omega_Z)^2\right] \\ &+ \omega(\omega - ik^2\nu_m)\,(\Omega^2{}_A + \Omega^2{}_B)\,\Omega^2{}_A - \Omega^4{}_A\,\Omega^2{}_J = 0 \end{aligned} \tag{4}$$

Where,

$$\Omega^2{}_A = \frac{k^2H^2{}_z}{4\pi\rho}$$

$$\Omega^2{}_B = \frac{k^2H^2{}_y}{4\pi\rho}$$

$$\Omega^2 = |\,\Omega\,|^2$$

$$\Omega^2{}_J = c^2k^2 - 4\pi G\rho$$

On rewriting (4) in descending powers of ω, we get

$$\begin{aligned} &\omega^6 - i\,(2F' + 3F)\,\omega^5 - (3F^2 + 6FF' + F'^2 + A + B)\,\omega^4 \\ &+ i\left[F^3 + 6F^2F' + 3FF'^2 + (A+2B)\,F + (2A + 2B - B')\,F'\right]\omega^3 \\ &+ \left[2F^3F' + 3F^2F'^2 + BF^2 + (A+B-B')\,F'^2 + 2\,(A+2B-B')\,FF' + C + D\right]\omega^2 \end{aligned}$$

$$-i\,[F^3F'^2 + F^2F'\,(2B-B') + FF'^2(A+2B-2B') + FD+F'(2C-C'+D-D')\,]\,\omega$$
$$-\,[F^2F'^2\,(B-B') + FF'(D-D') + F'^2\,(C-C') + E\,] = 0 \tag{5}$$

where,

$$\begin{aligned}
&A=4\Omega^2 && B=2\Omega^2_A + \Omega^2_B + \Omega^2_J\\
&C=4[\Omega^2_A\,\Omega^2_X + \Omega^2_B\,\Omega^2_J + (\Omega_A\,\Omega_Y - \Omega_B\,\Omega_Z\,)^2] && D=2\Omega^2_A\,\Omega^2_J + \Omega^2_A\,(\Omega^2_A + \Omega^2_B)\\
&E=\Omega^4_A\,\Omega^2_J && B'=2\Omega^2_A + \Omega^2_B\\
&C'=4[\Omega^2_A\,\Omega^2_X + (\Omega_A\,\Omega_Y - \Omega_B\,\Omega_Z\,)^2] && D'=\Omega^2_A\,(\Omega^2_A + \Omega^2_B\,)\\
&F=k^2\nu && F'=k^2\nu_m
\end{aligned} \tag{6}$$

The condition for stability is that the imaginary part of ω (all roots) should be positive since this ensures that the amplitude of the wave form does not grow with time. We put $\omega = -iz$. The equation transforms to

$$\begin{aligned}
&Z^6 + (2F' + 3F)\,Z^5 + (3F^2 + 6FF' + F'^2 + A+B)\,Z^4\\
&+[F^3 + 6F^2F' + 3FF'^2 + (A + 2B)F + (2A+2B-B')F']\,Z^3\\
&+[2F^3F' + 3F^2F'^2 + BF^2 + (A+B-B')F'^2 + 2(A+2B-B')\,FF' + C + D]Z^2\\
&+[F^3F'^2 + F^2F'(2B-B') + (A+2B-2B')FF'^2 + DF + (2C-C'+D-D')F']Z\\
&+[F^2F'^2\,(B-B') + FF'\,(D-D') + F'^2\,(C-C') + E] = 0
\end{aligned} \tag{7}$$

The stability condition is now that the real part of Z should be negative. The necessary and sufficient conditions for this to be so are given by the Routh-Hurmitz criterion which requires for a polynomial

$$a_0Z^n + a_1Z^{n-1} + \quad \cdots \quad \cdots \quad a_n = 0 \qquad \text{that}$$

$$D_1 = a_1 > 0$$

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0$$

$$\vdots$$

$$D_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ a_3 & a_2 & a_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2n} \end{vmatrix} > 0$$

where $a_m = 0$, $m > n$, n being the degree of the polynomial. In the nth determinant, the terms of the bottom row are all zero except a_n

$$D_n = a_n\,D_{n-1}$$

Since both $D_n > 0$ and $D_{n-1} > 0$ for stability, necessary condition for stability is $a_n > 0$. Applying this to the equation (7) we get

$$a_6 = [F^2F'^2\,(B-B') + F'^2\,(C-C') + FF'\,(D-D') + E]$$

$$= (F^2F'^2 + 4\Omega^2_z\, F'^2 + 2\Omega^2_A\, FF' + \Omega^4_A)\, \Omega^2_J > 0, \text{ if } \Omega^2_J > 0$$

Hence $\Omega^2_J > 0$ is a necessary condition. However, this condition is not sufficient and in addition we must have $D_1 > 0$, $D_2 > 0 \ldots\ldots D_n > 0$. Since F and F' are positive quantities $D_1 = 2F' + 3F > 0$ is automatically satisfied. The other four conditions give rise to complicated relations between the quantities A, B, C, F etc. If, however, $a_6 < 0$ so that $\Omega^2_J = C^2k^2 - 4\pi G\rho < 0$, the system is gravitationally unstable. This is precisely the Jeans criterion which remains uneffected by the combined effect of coriolis force and a magnetic field in a medium of finite viscosity and electrical conductivity.

SPECIAL CASES

We shall now discuss the necessary and sufficient condition for stability in two special cases.

Case I—The medium is supposed to be non-viscous and the rotation has only one component in the X—direction. $\Omega = (\Omega_x\, , 0, 0)$. In this case we have

$$a_0 = 1$$
$$a_1 = 2F'$$
$$a_2 = F'^2 + (4\Omega^2_X + 2\Omega^2_A + \Omega^2_J)$$
$$a_3 = F'\,(8\Omega^2_X + 2\Omega^2_A + \Omega^2_B + 2\Omega^2_J)$$
$$a_4 = (4\Omega^2_X + \Omega^2_J)\, F'^2 + (4\Omega^2_X + \Omega^2_A + \Omega^2_B + 2\Omega^2_J)\,\Omega^2_A$$
$$a_5 = F'\,(4\Omega^2_X + 2\Omega^2_J)\,\Omega^2_A$$
$$a_6 = \Omega^4_A\, \Omega^2_J$$

For stability $a_6 > 0$ which gives $\Omega^2_J > 0$ as a necessary condition, we get the values of $D_1, D_2 \ldots D_5$ as

$$D_1 = 2F' > 0$$
$$D_2 = F'\,(2F'^2 + 2\Omega^2_A + \Omega^2_B) > 0$$
$$D_3 = a_3\, D_2 - a_1\,(a_1\, a_4 - a_5)$$
$$= F'^2\,[\,(4\Omega^2_X + 2\Omega^2_B)\, F'^2 + 8\Omega^2_X\,(\Omega^2_A + \Omega^2_B) + \Omega^2_B\,(\Omega^2_B + 2\Omega^2_J)\,]$$
$$D_4 = a_4\, D_3 + (a_1\, a_6 - a_2\, a_5)\, D_2 + a_5\,(a_1\, a_4 - a_5)$$
$$D_5 = a^5\, D_4 - a_3\, a_6\, D_3 + a_1\, a_6\,(a_5\, D_2 - a_1^2\, a_6)$$

We have calculated D_4 and D_5 but the expressions are too lengthy to be mentioned here. $D_1, D_2 \ldots\ldots\ldots D_5$ are all > 0 when $\Omega^2_J > 0$. Therefore $\Omega^2_J > 0$ is both necessary and sufficient condition for stability of the medium.

Case II—We assume the rotation to be zero in this case. We have $A = C = C' = 0$

$$a_0 = 1$$
$$a_1 = 2F' + 3F$$
$$a_2 = (3F^2 + 6FF' + F'^2) + (2\Omega^2_A + \Omega^2_B + \Omega^2_J)$$
$$a_3 = (F^3 + 6F^2F' + 3FF'^2) + 2(2\Omega^2_A + \Omega^2_B + \Omega^2_J)\, F + (2\Omega^2_A + \Omega^2_B + 2\Omega^2_J)\, F'$$

$$a_4 = (2F^3F' + 3F^2F'^2) + (2\Omega^2_A + \Omega^2_B + \Omega^2_J)\, F^2 + \Omega^2_J\, F'^2$$
$$+ 2\,(2\Omega^2_A + \Omega^2_B + 2\Omega^2_J)\, FF' + \Omega^2_A\,(\Omega^2_A + \Omega^2_B + 2\Omega^2_J)$$
$$a_5 = F^3F'^2 + 2\Omega^2_J\, FF'^2 + (2\Omega^2_A + \Omega^2_B + 2\Omega^2_J)\, F^2F'$$
$$+ \Omega^2_A\,(\Omega^2_A + \Omega^2_B + 2\Omega^2_J)\, F + 2\Omega^2_A\, \Omega^2_J\, F'$$
$$a_6 = (F^2F'^2 + 2FF'\, \Omega^2_A + \Omega^4_A)\, \Omega^2_J$$

we find $a_6 > 0$ if $\Omega^2_J > 0$ so that $\Omega^2_J > 0$ is a necessary condition for stability. Solving the various determinants we get

$$D_1 = (2F' + 3F) > 0$$
$$D_2 = (8F^3 + 12F^2F' + 12FF'^2 + 2F'^3) + (2\Omega^2_A + \Omega^2_B + \Omega^2_J)F$$
$$+ (2\Omega^2_A\, \Omega^2_B)\, F'$$
$$D_2 > 0 \quad \text{if} \quad \Omega^2_J > 0$$

We have calculated D_3, D_4, D_5 also and they come out positive if $\Omega^2_J > 0$, expressions being too lengthy to be mentioned here.

So $\Omega^2_J > 0$ is both a necessary and sufficient condition for stability in the case of an irrotational flow in a medium of finite electrical conductivity and viscosity.

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