NON-UNIFORM NOZZLE FLOW—HEAT EXCHANGE EFFECTS

J. N. BERI

Scientific Analysis Group, New Delhi

(*Received* 10 *Oct.* 1964; *revised* 14 *Dec.* 1964)

Considering boundary layer effects in one-dimensional flow with heat exchange, true mean value flow parameters and their ratio to uniform free-stream values are derived. For one seventh power velocity distribution law and similar stagnation temperature and velocity profiles variations in Mach number, stagnation temperature and impulse at the exit section are plotted versus boundary layer thickness.

One dimensional theory of nozzle flow is based on the assumption that flow parameters are uniformly distributed over the cross-section of the duct or nozzle. Physically, flow is never uniform due to the presence of boundary layer region near the wall surface which causes errors in flow parameters. Tyler[1] has studied this non-uniformity of flow parameters under the assumption that both static pressure p and stagnation temperature T_0 are uniform across the section, the latter implying that there is no heat exchange between the working fluid and the surroundings. In particular, $1/n$ th power velocity distribution law (equation 1) across the duct section is used.

$$\left.\begin{aligned} \frac{q}{q'} &= \left(\frac{y}{\delta}\right)^{\frac{1}{n}} \qquad & o \leqslant y \leqslant \delta \\ &= 1 & \delta \leqslant y \leqslant R \end{aligned}\right\} \tag{1}$$

where the symbols have their usual meanings.

However, in divergent portion of the nozzle situated directly down-stream of combustion chamber, as in rockets and ram-jets, heat may be released to the working fluid due to inefficient or delayed combustion of propellant products during expansion process. In addition there may be heat transfer across the boundary layer (due to non-uniform flow) which accounts for the viscous and frictional losses in the boundary layer region. Both these processes cause dissipation of energy and a knowledge of their effects on flow parameters is of fundamental importance from the design as well as efficiency point of view.

MEAN VALUE FLOW

In non-uniform flow with heat exchange it is assumed that all the flow parameters except the pressure p are non-uniformly distributed over the cross-section and in particular, velocity is assumed to follow the distribution law given by equation (1). Under these conditions the principle of conservation of mass, momentum and energy applied to a single stream tube of area δA normal to the flow and summed over the whole cross-section gives

Continuity equation

$$\begin{aligned} \Sigma_1 \rho\, q\, \delta A &= \omega = \Sigma_2 \rho\, q\, \delta A \\ \text{or} \qquad \omega &= \bar{\rho}\, A\, \bar{q} \end{aligned} \tag{2}$$

Momentum equation

$X = \overline{F}_2 - \overline{F}_1$ with mean impulse given by

$$\overline{F} = Ap \left(1 + \gamma \frac{\xi_2 \eta}{\xi_3} \overline{M}^2 \right) \qquad (3)$$

Energy equation

$Q - W = C_p (\overline{T}_{o2} - \overline{T}_{o1})$, where the mean stagnation temperature $\overline{T}_o$ is given by

$$\overline{T}_o = \frac{1}{\omega} \sum \left(\frac{p\,q}{R} + \frac{\rho\, q^3}{2C_p} \right) \delta A$$
$$= \eta\, \overline{T} \left(1 + \frac{\gamma - 1}{2} \overline{M}^2 \right) \qquad (4)$$

and the mean Mach number $\overline{M}^2$ is given by

$$\overline{M}^2 = \frac{\xi_3 \quad \overline{q/s}^2}{\gamma\, R\, \eta\, \overline{T}} \qquad (5)$$

Equation of state

$$p = \overline{\rho}\, R\, \overline{T} \qquad (6)$$

The non-uniformities of flow parameters in these equations are approximated by suitable mean values and the correction factors ξ_i and η which take the value unity in free-stream and are defined thus

$$\Sigma\ \rho\, \overline{q}^i\, \delta A = \rho\, \overline{q}^i A, \qquad i = 0, 1, 2, 3 \qquad (7a)$$

$$\Sigma\ q\ \delta A = \eta\, q\, \overline{A} \qquad (7b)$$

$$\overline{q}^i = \xi_i\, \overline{q^i} \qquad (7c)$$

HEAT EXCHANGE EFFECTS

As mentioned earlier, heat transfer in the divergent nozzle is due to internal release of energy and the boundary layer effects. The process may thus be split up into two parts

(*a*) *Heat exchange in free-stream*—In free-stream where the flow is uniform this heat exchange is reflected through the variations of stagnation temperature T_o as

$$\frac{dT_o}{T_o} = \frac{2}{1 + \gamma M^2} \frac{dA}{A} + \frac{1 - M^2}{(1 + \gamma M^2) \left(1 + \frac{\gamma - 1}{2} M^2 \right)} \frac{dM^2}{M^2} \qquad (8)$$

Trends in nozzle performance parameters at each section could then be studied if it is assumed that the rate of heat exchange across the section is given by

$$\frac{A}{A_s} = \left(\frac{T_o}{T_{os}} \right)^k \qquad (9)$$

where 's' is some reference section and k is a parameter whose value is controlled by the temperature rise considered across any section. For sonic flow throughout $k=1\cdot 15$ $(\gamma=1\cdot 30)$ and for $k>1\cdot 15$ the flow is supersonic in the divergent stream. In particular $T_o/T_{os}=1$ corresponds to isentropic flow[2].

(b) *Heat transfer due to boundary layer*—For this purpose use is made of the similarity between the velocity and temperature profiles. In fully developed turbulent flow Reichardt[3] derived the temperature and velocity distribution relation given by

$$\frac{T}{T'}=\left(\frac{q}{q'}\right)^{P_t}, \quad P_t=\frac{A_T}{A_q} \tag{10}$$

where P_t is the ratio of momentum transfer coefficient to heat transfer coefficient and prime denotes maximum value in free-stream. In pipe flow Ludweig determined the value of this ratio as $A_q/A_T=1\cdot 15$ near the axis to $1\cdot 10$ near the wall[3].

Further in the laminar sub-layer of the compressible boundary layer it has been established that similarity prevails with regard to the longitudinal velocity components and as far as the temperature profiles are concerned it is observed that the profiles of stagnation enthalpy are identical with the velocity profiles. In this case, for zero pressure gradient across the boundary layer, velocity and temperature satisfy the simple relation

$$\frac{h-h_1}{h'-h_1}=\frac{q}{q'} \tag{11}$$

where h is the stagnation enthalpy ($h=C_p\,T_o$) and subscript (1) and prime (′) refer to wall and free-stream values respectively.

From these arguments it can justifiably be said that the stagnation temperature and velocity have similar profiles and follow the relation

$$\frac{T_o}{T_o'}=\left(\frac{q}{q'}\right)^m \tag{12}$$

This is the most general expression involving in itself equations (10) and (11) and the value of m is to be correlated through experimental observations. $m=1$ corresponds to identical profiles of stagnation enthalpy and velocity orientated by a scale transformation.

EXPRESSION FOR ξ_i AND η

In order to determine the errors in flow parameters it is necessary to evaluate correction factors, ξ_i and η from equations (7) and for this by defining

$$Y_i=\sum\frac{\rho}{\rho'}\left(\frac{q}{q'}\right)^i\frac{\delta A}{A} \tag{13}$$

we get

$$\xi_i=\left(\frac{Y_0}{Y_1}\right)^i\frac{Y_i}{Y_o} \tag{14}$$

Making use of the assumptions of uniformity of pressure over the cross-section, the non-uniform velocity distribution equation (1) and the stagnation temperature and velocity distribution relation equation (12) it is simple to deduce the following expressions for Y_i a d η

$$Y_i=1\quad 2\frac{\delta}{R}\left(1-\frac{Z_i}{1+\frac{\gamma-1}{2}M'^2}\right)+\left(\frac{\delta}{R}\right)^2\left(1-\frac{2\,Z_{i+n}}{1+\frac{\gamma-1}{2}M'^2}\right) \tag{15}$$

where

$$\xi_i = \sum_{j=o}^{\infty} \left(\frac{\frac{\gamma-1}{2} M'^2}{1+\frac{\gamma-1}{2} M'^2} \right)^j \frac{n}{(i+n-m)+j(2-m)} \tag{16}$$

and

$$\eta = \frac{Y_o}{Y_1} \left[1 - \frac{2}{n+1} \frac{\delta}{R} + \frac{1}{2n+1} \left(\frac{\delta}{R} \right)^2 \right] \tag{17}$$

GENERAL RELATIONS

Before we proceed to find general expressions for one-dimensional non-uniform flow parameters, it is essential to deduce the central parameter stagnation temperature $\overline{T_o}$ of the problem in terms of free-stream Mach number M' and correction factors ξ_i and η. From the 1st of equation (4) we have

$$\omega \; \overline{T}_o = \sum \left(\frac{pq}{R} + \frac{\rho q^3}{2\,C_p} \right) \delta A$$

Substituting for $\frac{p}{R}$ and ω and simplifying with the help of equation (13), we get

$$\overline{T}_o = \frac{T'}{Y_1} \left[\eta \frac{\overline{q}}{q'} + \frac{\gamma-1}{2} M'^2 Y_3 \right]$$

Further since $\frac{\overline{q}}{q'} = \frac{Y_1}{Y_o}$, from equations (7), (13) and $T'_o = T' \left(1 + \frac{\gamma-1}{2} M'^2 \right)$ we have

$$\frac{\overline{T}_o}{T'_o} = \frac{\eta \frac{\overline{q}}{q'} + \frac{\gamma-1}{2} M'^2 \; Y_3}{Y_1 \left(1 + \frac{\gamma-1}{2} M'^2 \right)}$$

$$= \frac{\eta \frac{Y_1}{Y_o} + \frac{\gamma-1}{1} M'^2 Y_3}{Y_1 \left(1 + \frac{\gamma-1}{2} M'^2 \right)} \tag{18}$$

Now taking into account the heat exchange effects in free-stream flow, equations (8) and (9) give

$$\frac{\rho' q'}{\rho_s \; q_s} = \left[M'^2 \left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2} M'^2} \right)^{2x-1} \left(\frac{1-2k+\gamma}{1-2k+\gamma M'^2} \right)^{2(1-x)} \right]^{\frac{-k}{1-2k}} \tag{19}$$

where

$$x = \frac{2\,k\,\gamma}{(\gamma+1) + 2k\,(\gamma-1)} \tag{20}$$

and subscript 's' refers to initially sonic flow at throat. Considering the non-uniformities of flow parameters due to boundary layer effects the principle of conservation of mass equation (2) and the mean value flow parameters as defined in the preceding sections, we have for initial sonic flow at the nozzle throat

$$\left[\frac{\overline{(A)}}{A_s} \right]^{-1} = \frac{\overline{\rho}\ \overline{q}}{\rho_s\ q_s} = Y_1 \ \frac{\rho'\ q'}{\rho_s\ q_s} \tag{21}$$

and on using equation (19) we get the following relation between mean Mach number and nozzle flow area

$$\frac{\overline{A}}{A_s} = \frac{1}{Y_1} \left[\frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{Y_2} \left(\frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2} \frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{Y^2}} \right)^{2x-1} \left(\frac{1-2\,k+\gamma}{1-2k + \gamma \frac{\xi_2\eta}{\xi_3} \frac{\overline{M}^2}{Y^2}} \right)^{2(1-x)} \right]^{\frac{k}{1-2k}} \tag{22}$$

the free-stream Mach number being related to mean Mach number. Thus

$$M^2 = \frac{\xi_3}{\eta} \left(\frac{Y_1}{Y_o} \right)^2 Y_o\ M'^2 \tag{23}$$

Further, eliminating $\eta\,\overline{T}$ from equation (3) and (4) and taking the ratio to sonic values, the mean velocity ratio is

$$\frac{\overline{q}}{q_s} = \sqrt{\frac{\frac{\gamma+1}{2}\ \overline{M}^2}{\xi_3\left(1 + \frac{\gamma-1}{2}\ \overline{M}^2\right)}} \sqrt{\frac{\overline{T}_o}{T_{os}}} \tag{24}$$

Similarly, ratios for other mean value parameters can now be obtained in terms of cross-sectional area and stagnation temperature. The corresponding values for uniform flow with heat exchange are immediately obtained by substituting $\eta = \xi_i = 1$.

However, since our purpose is to find the errors due to the non-uniformity of flow across the section, we finally deduce ratios of mean value flow parameters to the corresponding parameters of uniform flow in terms of mean value Mach number and boundary layer correction factors. Thus

Velocity ratio

$$\frac{\overline{q}}{q'} = \frac{Y_1}{\sqrt{\eta\, Y_o}} \sqrt{\frac{1 + \frac{\gamma-1}{2} \frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{Y_2}}{1 + \frac{\gamma-1}{2}\ \overline{M}^2}} \sqrt{\frac{\overline{T}_o}{T'_o}} \tag{25}$$

Pressure ratio

$$\frac{\overline{(p)}}{p'} = \sqrt{\frac{Y_o}{\eta}} \sqrt{\frac{1+ \frac{\gamma-1}{2} \frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{Y_2}}{1+ \frac{\gamma-1}{2} \overline{M}^2}} \sqrt{\frac{\overline{T}_o}{T'_o}} \tag{26}$$

Impulse ratio

$$\frac{\overline{F}}{\overline{F}'} = \frac{1+\gamma \frac{\xi_2\,\eta}{\xi_3} \overline{M}^2}{Y_1\left(1 + \gamma \frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{Y_2}\right)} \sqrt{\frac{Y_o \left(1+ \frac{\gamma-1}{2} \frac{\xi_2\,\eta}{\xi_3} \frac{\overline{M}^2}{y_2}\right)}{\eta\ \left(1+ \frac{\gamma-1}{2}\ \overline{M}^2\right)}} \sqrt{\frac{T_o}{T'_o}} \tag{27}$$

where $\bar{T}_o/T'_o$ is given by equation (18). Ratios for other parameters may also be found similarly.

Application to problems

Two sets of equations have been obtained above. The first set equations (21) to (24) enables us to have calculations corresponding to one dimensional formula with heat exchange, but taking due account of non-uniform velocity distribution. The origin of such flows could be taken either a stagnation region or a region where the boundary layer thickness is zero. The second set of equations (25) to (27) permits us to assess the accuracy of one dimensional results. In evaluating these errors caused by the presence of boundary layer region in one dimensional flow problems involving heat exchange, care has to be taken in the use of a proper mean-value equation corresponding to given boundary conditions.

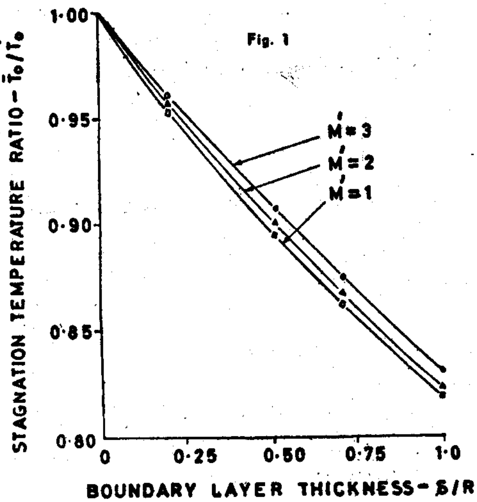

FIG. 1—Variation of stagnation temperature ratio with boundary layer thickness.

For simplicity, we consider the case where the average rate of boundary layer heat transfer across the originally known free-stream flow is given. Then, the errors that are likely to be caused in the exit Mach number [equation (23)] and Impulse function [equation (27)] corresponding to the temperature variations of equation (18) *versus* boundary layer thickness are graphically presented in Figs. 1-3.

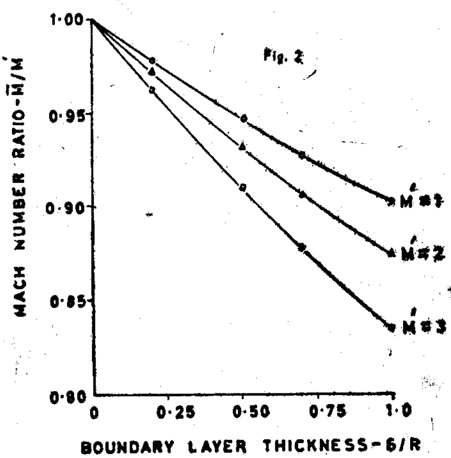

FIG. 2—Variation of Mach number ratio with boundary layer thickness.

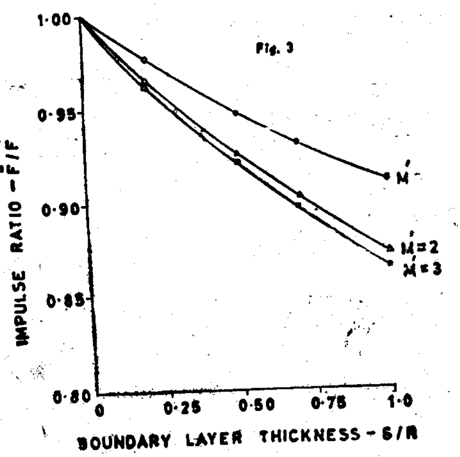

FIG. 3—Variation of impulse ratio with boundary layer thickness.

These curves which present ratios of mean value solutions to one dimensional uniform flow solution indicate directly the error caused in flow parameters due to the neglect of the existence of boundary layer region. In general it could be inferred that the error tends to increase with Mach number, and also with boundary layer thickness except for the total temperature ratio, in which case though the error increases with boundary layer thickness, this increase decreases with higher Mach number. It may be seen that upto boundary layer thickness of 20 per cent : for $M'=1$ the error in Mach number is about 2 per cent and that in Impulse $2\frac{1}{4}$ per cent corresponding to the temperature ratio error of $4\frac{3}{4}$ per cent; for $M'=2$ whereas the error in Mach number and impulse increases to about $2\frac{3}{4}$ per cent and $3\frac{3}{4}$ per cent respectively the error in temperature ratio decreases to $4\frac{1}{4}$ per cent; and for $M'=3$ the error in Mach number is about $3\frac{3}{4}$ per cent, that in impulse ratio remains stationary at $3\frac{3}{4}$ per cent and also the decrease in temperature ratio is only 4 per cent. This indicates that as the exit speed increases beyond a certain limit no further errors are added. Further, for thicker boundary layers the errors increase almost nearly in proportion.

ACKNOWLEDGEMENTS

The author is grateful to Dr. V.R. Thiruvenkatachar, F.N.I. for his helpful guidance in the analysis of this paper and to Shri P.C. Manglik for doing the calculation work.

REFERENCES

1. TYLER, R.D., "Aeronautical Research Council Technical Report R&M No. 2991" (H.M.S.O., London), 1957.

2. HEARTH, D.P. & PERCHONOK, E., "Analysis of Heat Addition in a Convergent-Divergent Nozzle—Report No. 2938" (N.A.C.A., Washington), 1953.

3. SCHLICHTING, HERMANN, "Boundary Layer Theory" (McGraw Hill Book Inc., New York) 1960, p. 611.