A BIVARIATE GAMMA DISTRIBUTION IN LIFE TESTING

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A bivariate gamma distribution as a model in life testing problem has been proposed and various statistical properties are studied.

Freund[1] has considered a bivariate extension of the exponential distribution proposed as a model in life testing for two component systems and more recently following a similar approach, the authors[2] have considered a further generalisation to obtain bivariate models of life time for such systems. In this paper, the authors consider a further extension of the problem and obtain a bivariate gamma distribution as a model in life testing. Bivariate and multivariate extensions of gamma distribution already existing in literature [3,4] have been derived in a manner such that the marginal distributions reduce to the same form as the parent population. The distribution considered in this paper is, however, obtained bearing the problem of life testing specifically in mind and it is found that the marginal distributions are not of gamma type. The study of the statistical properties of the proposed model have also been considered. The results for the general case seem to be intractable and therefore, in view of the complexity involved, suitable approximations have been taken.

DERIVATION OF THE MODEL

We assume that the random variables X and Y represent the life times of two components A and B respectively in a two component system, X^* represents the life time of the component A if the component B is replaced with the component of the same kind each time it fails (if necessary more than once), and Y^* represents the life time of the component B if the component A is replaced with a component of the same kind each time it fails (if necessary more than once). We also assume that X^* and Y^* are independently distributed gamma variates given below:

$$f(x^*) = \frac{e^{-x^*/\alpha}\left(\frac{x^*}{\alpha}\right)^{p-1}}{\Gamma(p)\,\alpha}, \qquad \begin{matrix}\alpha,\ p > 0 \\ (0 < x^* < \infty)\end{matrix} \tag{1}$$

$$f(y^*) = \frac{e^{-y^*/\beta}\left(\frac{y^*}{\beta}\right)^{q-1}}{\Gamma(q)\,\beta}, \qquad \begin{matrix}\beta, q > 0 \\ (0 < y^* < \infty)\end{matrix} \tag{2}$$

Under these assumptions the element of probability that the component A fails first at time x^* and that the component B has not yet failed is

$$[\, f(x^*)\, dx^*]\ [\, 1 - I_{x^*}(\beta,\ q)\,] \tag{3}$$

and the element of probability that the component B fails first at y^* and A has not yet failed is

$$[\, f(y^*)\, dy^*]\ [\, 1 - I_{y^*}(\alpha,\ p)\,] \tag{4}$$

where

$$I_{x^*} \ (\beta, q) = \int_0^{x^*} f\ (y^*)\ dy^* \tag{5}$$

and

$$Iy^* \ (\alpha, p) = \int_0^{y^*} f\ (x^*)\ dx^* \tag{6}$$

Furthermore, the conditional probability density of Y given that the component A fails at x^* is

$$g(y^*) = \frac{f\ (y^*)}{[\ 1 - I_{x^*} \ (\beta, q)\ ]}, \qquad (0 < x^* < y^* < \infty)$$

$$= 0 \text{ otherwise} \tag{7}$$

and similarly the conditional probability density of X given that the component B fails first at y^* is

$$g(x^*) = \frac{f\ (x^*)}{[\ 1 - Iy^* \ (\alpha, p)\ ]}, \qquad (0 < y^* < x^* < \infty)$$

$$= 0 \quad \text{otherwise.} \tag{8}$$

Considering now the case where the components are not replaced the element of probability that the component A fails first at x and that the component B has not yet failed is

$$[f(x)\ dx\ ]\ [\ 1 - I_x\ (\ \beta,\ q\ )\ ] \tag{9}$$

analogous to (3) and the element of probability that the component B fails first at y and that A has not yet failed is

$$[f(y)\ \ dy\ ]\ [\ 1 - I_y\ (\ \alpha,\ p\ )\ ] \tag{10}$$

analogous to (4), where $f(x)$, $f(y)$, $Ix\ (\beta, q)$, and $I_y\ (\ \alpha,\ p\ )$ are given by (1), (2), (5) and (6) respectively, with stars(*) omitted.

In order to completely specify the bivariate model under consideration we assume that when the component B fails, the system still works but the conditional probability density of X given that the component B fails first at y is of the same form as (8) though the parameters change from p and α to p' and α' respectively, that is,

$$g\ (x) = \frac{f^*(x)}{[\ 1 - Iy_*\ (\alpha', p')\ ]}, \qquad (\ 0 < y < x < \infty)$$

$$= 0 \quad \text{otherwise} \tag{11}$$

where

$$f^*\ (x) = \frac{e^{-x/\alpha'}\ (x/\alpha')^{p'-1}}{\alpha'\ \Gamma(p')}, \qquad (\ \alpha' > 0,\ p' > 0,\ 0 < x < \infty) \tag{12}$$

and

$$I_{y^*}(\alpha', p') = \int_0^y f^*(x)\, dx \tag{13}$$

Similarly the conditional probability density of Y given that the component A fails first at x is,

$$g(y) = \frac{f^*(y)}{[1 - I_{x^*}(\beta', q')]} \; ; \qquad (0 < x < y < \infty)$$

$$= 0 \text{ otherwise} \tag{14}$$

where

$$f^*(y) = \frac{e^{-y/\beta'} (y/\beta')^{q'-1}}{\beta' \Gamma(q')}, \qquad (\beta' > 0, q' > 0, 0 < y < \infty) \tag{15}$$

and

$$I_{x^*}(\beta', q') = \int_0^x f^*(y)\, dy \tag{16}$$

It now easily follows from these assumptions that the joint density of X and Y is

$$f(x, y) = \begin{cases} \dfrac{f(x) f^*(y) \{1 - I_x(\beta, q)\}}{\{1 - I_x^*(\beta', q')\}}, & (0 < x < y < \infty) \\[2ex] \dfrac{f^*(x) f(y) \{1 - I_y(\alpha, p)\}}{\{1 - I_y^*(\alpha', p')\}}, & (0 < y < x < \infty) \end{cases} \tag{17}$$

This p.d.f. may alternatively be expressed in the form given below:

$$f(x, y) = \begin{matrix} \phi(x) f^*(y) & (0 < x < y < \infty) \\ f^*(x) \Psi(y) & (0 < y < x < \infty) \end{matrix} \tag{18}$$

where

$$\phi(x) = f(x) \frac{\left\{ 1 - (x/\beta)^q \cdot \dfrac{1}{\Gamma(q+1)} \; {}_1F_1\left[q; q+1; -\dfrac{x}{\beta}\right] \right\}}{\left\{ 1 - (x/\beta')^{q'} \cdot \dfrac{1}{\Gamma(q'+1)} \; {}_1F_1\left[q'; q'+1; -\dfrac{x}{\beta'}\right] \right\}}$$

$$\Psi(y) = f(y) \left\{ \frac{1 - (y/\alpha)^p \cdot \dfrac{1}{\Gamma(p+1)} \; {}_1F_1[p; p+1; -y/\alpha]}{1 - (y/\alpha')^{p'} \cdot \dfrac{1}{\Gamma(p'+1)} \; {}_1F_1[p'; p'+1; -y/\alpha']} \right\}$$

and

$${}_1F_1[a; b; x] = \sum_{n=0}^{\infty} \frac{(a)n}{(b)n} \cdot \frac{x^n}{n!}$$

is the well-known Kummer's function[5].

For the particular case $p = p' = q = q' = 1$, it is easily seen that the p.d.f. (18) reduces to the model of Freund[1].

STATISTICAL PROPERTIES OF THE PROPOSED MODELS

The evaluation of the integrals involved in the general expression for the moments seems to be extremely difficult and therefore two cases imposing certain restrictions on the parametric values have been discussed separately.

Case I : p, q, p' and q' large.

This imposes certain restrictions on the shape of the distributions (1) and (2) and we shall first confine our attention to this case. We have for (r, s)th moment the expression

$$\mu_{rs} = \iint x^r \, y^s \, f(x, y) \, dx \, dy \tag{19}$$

where the integration is to be carried over the appropriate regions as given by (18).

It is well known that for large values of a and b such that $(b-a)$ remains bounded, we have for a given value x the approximation

$${}_1F_1 [\, a; b; x \,] \simeq e^x \, [\, 1 + 0 (\, | \, b \, |^{-1} \,) \,]$$

and we use this fact in evaluating

$$\mu_{rs} = \int_0^\infty y^s f^*(y) \left[\int_0^y x^r \phi(x) \, dx \right] dy + \int_0^\infty x^r f^*(x) \left[\int_0^x y^s \Psi(y) \, dy \right] dx$$

Remembering the fact that p, q etc. are large so that the second and higher orders of $\dfrac{1}{\Gamma(q+1)}$ etc. could be neglected the last integral yields the following result for the moments of the distribution:

$$\left.\begin{aligned}
\mu_{rs} = {} & \frac{\Gamma(r+s+p+q')}{\Gamma(p)\,\Gamma(q')\,\alpha^p\,(\beta')^{q'}\,(r+p)} \left[\frac{\alpha\beta'}{\alpha+\beta'}\right]^{r+s+p+q'} \\
& + \frac{\Gamma(r+s+p+2q')}{\Gamma(p)\,\Gamma(q')\,\Gamma(q'+1)\alpha^p\,(\beta')^{2q'}\,(r+p+q')} \left[\frac{\alpha\beta'}{2\alpha+\beta'}\right]^{r+s+p+2q'} \\
& + \frac{\Gamma(r+s+p'+q)}{\Gamma(q)\,\Gamma(p')\,\beta^q\,(\alpha')^{p'}\,(s+q)} \left[\frac{\alpha'\beta}{\alpha'+\beta}\right]^{r+s+p'+q} \\
& + \frac{\Gamma(r+s+2p'+q)}{\Gamma(p')\,\Gamma(q)\,\Gamma(p'+1)\beta^q\,(\alpha')^{2p'}\,(s+p'+q)} \left[\frac{\alpha'\beta}{\alpha'+2\beta}\right]^{r+s+2p'+q} \\
& - \frac{\Gamma(r+s+p+q+q')}{\Gamma(p)\,\Gamma(q')\,\Gamma(q+1)\alpha^p\,\beta^q\,(\beta')^{q'}\,(r+p+q)} \left[\frac{\alpha\beta\beta'}{\alpha\beta+\alpha\beta'+\beta\beta'}\right]^{r+s+p+q+q'} \\
& - \frac{\Gamma(r+s+p+q+p')}{\Gamma(p')\,\Gamma(q)\,(p+1)\alpha^p\,\beta^q\,(\alpha')^{p'}\,(p+q+s)} \left[\frac{\alpha\alpha'\beta}{\alpha\alpha'+\alpha'\beta+\alpha\beta}\right]^{r+s+p+q+p'}
\end{aligned}\right\} \tag{20}$$

Marginal densities: The marginal distribution of x is given by

$$F(x) = f^*(x) \int_0^x \psi(y)\,dy + \phi(x) \int_x^\infty f^*(y)\,dy \tag{21}$$

Using the approximations mentioned earlier, we get

$$F(x) = f^*(x) \left\{ \left(\frac{x}{\beta}\right)^q \frac{1}{\Gamma(q+1)}\, {}_1F_1[q; q+1; -x/\beta] + \frac{x^{p'+q}\, {}_1F_1\left[p'+q; p'+q+1; -\frac{\alpha'+\beta}{\alpha'\beta}.x\right]}{(p'+q)\,\beta^q\,(\alpha')^{p'}\,\Gamma(q)\,\Gamma(p'+1)} - \frac{x^{p+q}\, {}_1F_1\left[p+q; p+q+1; -\frac{\alpha+\beta}{\alpha\beta}.x\right]}{(p+q)\,\beta^q\,\alpha^p\,\Gamma(q)\,\Gamma(p+1)} \right\}$$

$$+ f(x)\left[1-(x/\beta)^q \frac{1}{\Gamma(q+1)}\,{}_1F_1[q; q+1; -x/\beta]\right] \tag{22}$$

In a similar manner we get the marginal densities of y thus:

$$G(y) = f^*(y) \left\{ \frac{\left(\frac{y}{\alpha}\right)^p}{\Gamma(p+1)}\,{}_1F_1[p; p+1; -y/\alpha] + \frac{y^{p+q'}\,{}_1F_1\left[p+q'; p+q'+1; -\frac{\alpha+\beta'}{\alpha\beta'}\,y\right]}{(p+q')\,\alpha^p\,(\beta')^{q'}\,\Gamma(p)\,\Gamma(q'+1)} - \frac{y^{p+q}\,{}_1F_1\left[p+q; p+q+1; -\frac{\alpha+\beta}{\alpha\beta}\,y\right]}{(p+q)\,\alpha^p\,\beta^q\,\Gamma(p)\,\Gamma(q+1)} \right\}$$

$$+ f(y)\left[1-\left(\frac{y}{\alpha}\right)^p . \frac{1}{\Gamma(p+1)}\,{}_1F_1[p; p+1; -y/\alpha]\right] \tag{23}$$

Thus we see that the marginal densities are not of gamma type unlike the extension proposed earlier.

Conditional mean : Obtaining the conditional density $f(y/x)$ by dividing the bivariate density by the marginal density of x it can be seen that

$$F(x)\,E(y/x) = \phi(x) \int_x^\infty y\,f^*(y)\,dy + f^*(x) \int_0^x y\,\psi(y)\,dy$$

$$= \phi(x)\, \beta'\, q' \left\{ 1 - \left(\frac{x}{\beta'}\right)^{q'+1} \frac{1}{\Gamma(q'+2)}\, {}_1F_1 \left[q' + 1; q'+2; \frac{-x}{\beta'} \right] \right\}$$

$$+ f^*(x) \left\{ \frac{\beta\, q\, (x/\beta)^{q+1}}{\Gamma(q+2)}\; {}_1F_1 \left[q+1; q+2; \frac{-x}{\beta} \right] \right. \qquad (24)$$

$$+ \frac{x^{p'+q+1}\; {}_1F_1 \left[p' + q+1; p'+q+2; \frac{-(\alpha'+\beta)}{\alpha'\,\beta} x \right]}{(p'+q+1)\, \Gamma(p'+1)\, \Gamma(q)\, (\alpha')^{p'} \beta^q}$$

$$\left. - \frac{x^{p+q+1}\; {}_1F_1 \left[p+q+1; p+q+2; \frac{-\alpha+\beta}{\alpha\,\beta} x \right]}{(p+q+1)\, \Gamma(p+1)\, \Gamma(q)\, \alpha^p \beta^q} \right\}$$

Thus the regression of y on x is not linear. In a similar manner it can be easily shown that the other regression equation is also not linear.

Case II : p, q, p' and q' very small

This again imposes certain restrictions on the shape of the distribution. But if the mean life times of the two components are not very small then naturally α and β will be moderately large. Further we assume that when a component fails, the mean life time of the other component working singly reduces but not drastically so that α' and β' are also large. However, we assume that $\alpha' < \beta$ and $\beta' < \alpha$ which may give a physical interpretation that the original mean life times are greater than the reduced mean life times. Now the moments of the bivariate distribution may again be obtained by integration of (18) over appropriate regions. Under the assumptions mentioned above, for a given x we have used the approximation

$${}_1F_1 \left[q; q+1; \frac{-x}{\beta} \right] \simeq \left[1 - \frac{x}{\beta} \cdot \frac{q}{q+1} \right]$$

and have neglected the terms of the second and higher orders of smallness. Thus carrying out the integration (18), we get

$$\mu_{rs} = \frac{\Gamma(r+s+p+q')\, (\beta')^{r+s+p}}{\Gamma(p)\, \Gamma(q')\, \alpha^p\, (r+p)}\; {}_2F_1 \left[r+p; r+s+p+q'; r+p+1; -\frac{\beta'}{\alpha} \right]$$

$$+ \frac{\Gamma(r+s+p+q+q'+1)\, q\, (\beta')^{r+s+p+q+1}}{\Gamma(p)\, \Gamma(q')\, \Gamma(q+2)\, \alpha^p\, \beta^{q+1}\, (r+p+q+1)} \times$$

$${}_2F_1 \left[r+p+q+1; r+p+q+q'+s+1; r+p+q+2; -\frac{\beta'}{\alpha} \right]$$

$$- \frac{\Gamma(r+s+p+q+q')\, (\beta')^{r+s+p+q}}{\alpha^p\, \beta^q\, \Gamma(p)\, \Gamma(q+1)\, \Gamma(q')\, (r+p+q)} \times$$

$${}_2F_1 \left[r+p+q; r+s+p+q+q'; r+p+q+1; -\frac{\beta'}{\alpha} \right]$$

$$+ \frac{\Gamma(r+p+s+2q')\, (\beta')^{r+s+p}}{\Gamma(p)\, \Gamma(q')\, \Gamma(q'+1)\, \alpha^p\, (r+p+q')} \times$$

$$ {}_2F_1\left[r+p+q';\ r+s+p+2q';\ r+p+q'+1;-\frac{\beta'}{\alpha}\right] $$

$$ -\frac{\Gamma(r+s+p+2q'+1)\,(\beta')^{r+s+p}}{\Gamma(p)\,\Gamma(q')\,\Gamma(q'+2)\,\alpha^{p}\,(r+p+q'+1)}\times $$

$$ {}_2F_1\left[r+p+q'+1;\ r+s+p+2q'+1;\ r+p+q'+2;-\frac{\beta'}{\alpha}\right] $$

$$ +\frac{\Gamma(r+s+p'+q)\,(\alpha')^{r+s+q}}{\Gamma(p')\,\Gamma(q)\,\beta^{q}\,(s+q)}\times $$

$$ {}_2F_1\left[q+s;\ q+p'+r+s;\ q+s+1;-\frac{\alpha'}{\beta}\right] $$

$$ +\frac{\Gamma(r+s+p+p'+q+1)\,p\,(\alpha')^{r+s+p+q+1}}{\Gamma(p')\,\Gamma(q)\,\Gamma(p+2)\,\alpha^{p+1}\,\beta^{q}\,(p+q+s+1)}\times $$

$$ {}_2F_1\left[p+q+s+1;\ p+q+r+s+p'+1;\ p+q+s+2;-\frac{\alpha'}{\beta}\right] $$

$$ -\frac{\Gamma(r+s+p+p'+q)\,(\alpha')^{r+s+p+q}}{\Gamma(p')\,\Gamma(q)\,\Gamma(p+1)\,\alpha^{p}\,\beta^{q}\,(s+p+q)}\times $$

$$ {}_2F_1\left[s+p+q;\ r+s+p+q+p';\ s+p+q+1;-\frac{\alpha'}{\beta}\right] $$

$$ +\frac{\Gamma(r+s+2p'+q)\,(\alpha')^{r+s+q}}{\Gamma(p')\,\Gamma(q)\,\Gamma(p'+1)\,\beta^{q}\,(s+p'+q)}\times $$

$$ {}_2F_1\left[s+p'+q;\ r+s+q+2p';\ s+p'+q+1;-\frac{\alpha'}{\beta}\right] $$

$$ -\frac{\Gamma(r+s+q+2p'+1)\,p'\,(\alpha')^{r+s+q}}{\Gamma(p^{1})\,\Gamma(q)\,\Gamma(p'+2)\,\beta^{q}\,(s+p'+q+1)}\times $$

$$ {}_2F_1\left[s+p'+q+1;\ r+s+2p'+q+1;\ s+p'+q+2;-\frac{\alpha'}{\beta}\right] \qquad (25) $$

where

${}_2F_1\,[a\,;\,b\,;\,c\,;\,x]$ is the well known Gauss function defined by

$$ {}_2F_1\,[a; b; c; x] = \sum_{n=o}^{\infty} \frac{(a)n\;\;(b)n}{(c)n}\;\frac{x^n}{n\,!} $$

the series being convergent whenever $|x| < 1$ and when $x=1$ provided that $Rl(c-a-b) > 0$ and when $x= -1$ provided that $Rl\ (c-a-b) > -1$.

Thus under the assumptions mentioned earlier, the moments are given by (25). For $\beta' > \alpha$ or $\alpha' > \beta$, obviously no moments exist. It may be easily seen that, for the limiting case $\alpha' = \beta$ as well as $\beta' = \alpha$, no moments exist.

Marginal densities : Marginal distribution of x is again given by (21). Using the approximations discussed for the case II and carrying out the integration we get

$$ F\,(x) = f\,(x)\;\left[\,1-(\,x/\beta)^q\;\frac{1}{\lceil(q+1)}\;{}_1F_1\,[q;\,q+1;\,-x/\beta]\,\right] $$

$$ + f^*\,(x)\;\left\{\;\frac{\left(\dfrac{x}{\beta}\right)^q}{\lceil(q+1)}\;{}_1F_1\,[q;\,q+1;\,-\,x/\beta] \right. $$

$$ + \frac{(\beta)^{p'}\;\left(\dfrac{x}{\beta}\right)^{p'+q}}{\lceil(p'+1)\;\lceil(q)\,(\alpha')^{p'}\,(p'+q)}\;{}_1F_1\,[p'+q;\,p'+q+1;\,-x/\beta] $$

$$ - \frac{p'\,(\beta)^{p'+1}\;\left(\dfrac{x}{\beta}\right)^{p'+q+1}}{\lceil(p'+2)\;\lceil(q)\,(\alpha')^{p'+1}\,(p'+q+1)}\;{}_1F_1\,[p'+q+1;\,p'+q+2;\,-x/\beta] $$

$$ - \frac{(\beta)^{p}\;\left(\dfrac{x}{\beta}\right)^{p+q}}{\lceil(p+1)\,\alpha^p\;\lceil(q)\,(p+q)}\;{}_1F_1\,[p+q;\,p+q+1;\,-\,x/\beta] $$

$$ \left. + \frac{\beta^{p+1}\;\left(\dfrac{x}{\beta}\right)^{p+q+1}}{\lceil(p+1)\;\lceil(q)\,\alpha^{p+1}\,(p+q+1)}\;{}_1F_1\!\left(p+q+1;\;p+q+2\,;\;-\frac{x}{\beta}\right)\right\} \qquad (26) $$

In a similar manner the marginal density $G(y)$ may also be calculated and be seen that it is not of gamma type.

Conditional mean :

Proceeding in the same manner as in case I, we get

$$ F\,(x).\,\overline{E}\,(y/x) = \phi\,(x)\,\beta^1\,q^1\left\{1-\frac{(x/\beta)^{q'+1}}{\lceil(q'+2)}\;{}_1F_1\left[q'+1;\;q'+\;2;\;-\;\frac{x}{\beta}\right]\right\} $$

$$+ f^*(x) \left\{ \frac{(\beta q)\left(\frac{x}{\beta}\right)^{q+1}}{\Gamma(q+2)} \, {}_1F_1\left[q+1; q+2; -\frac{x}{\beta}\right] \right.$$

$$+ (x)^{p'+q+1} \frac{{}_1F_1\left[p^1+q+1; p'+q+2; -\frac{x}{\beta}\right]}{(p'+q+1)\beta^q \Gamma(p'+1) \Gamma(q) (\alpha')^{p'}}$$

$$- p' x^{p'+q+2} \frac{{}_1F_1\left[p'+q+2; p'+q+3; -\frac{x}{\beta}\right]}{(p'+q+2)(\alpha')^{p'+q} \beta \Gamma(q) \Gamma(p'+2)}$$

$$- x^{p+q+1} \frac{{}_1F_1\left[p+q+1; p+q+2; -\frac{x}{\beta}\right]}{(p+q+1)\beta^q \alpha^p \Gamma q \Gamma(p+1)}$$

$$\left. + p x^{p+q+2} \frac{{}_1F_1\left[p+q+2; p+q+3; -\frac{x}{\beta}\right]}{(p+q+2)\beta^q \alpha^{p+1} \Gamma(p+2) \Gamma(q)} \right\} \quad (27)$$

In a similar manner the conditional mean $E(x/y)$ may also be worked out. Here also it may be observed that the regression equations are generally nonlinear.

It may also be remarked that for the particular case $p = p'$ and $\alpha = \alpha'$, the marginal densities $F(x)$ obtained for Case I and II as given by (22) and (26) respectively reduce to $f(x)$ as given in (1). However, this does not provide any information about the independence of the life of the component A from that of component of B. In fact it does not follow that $f(y/x) = f(y)$ in this case as might have been expected. But if we take $q = q'$ and $\beta = \beta'$ in addition to $p = p'$ and $\alpha = \alpha'$ it may be easily seen that $f(y/x) = f(y)$ and thus the life times of A and B are independent. Analogous arguments, of course, apply for the marginal density $G(y)$.

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