SHEAR FLOW OF AN ELASTICO-VISCOUS FLUID PAST A POROUS FLAT PLATE

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Considering shear flow of an elastico-viscous fluid past a porous flat plate, it has been found that a steady solution for the velocity field is not possible if there is fluid injection at the plate and that the vorticity in the shear flow increases the skin-friction at the plate. If the wall is impermeable, the elastic elements do not affect the velocity field. The thickness of the boundary layer on the plate decreases w ith the increase of relaxation time, but it increases with the increase of retardation time.

A problem that has recently stimulated much interest concerns the determination of the flow field about a body that is immersed in the stream of a viscous liquid that contains vorticity generated by some external mechanism other than the body. To study this problem in its essential features Li[1] introduced the idealized model of the two-dimensional, unbounded, steady, constant shear flow of an incompressible viscous fluid past an infinitesimally thin, semi-infinite flat plate that is aligned parallel to the oncoming flow. This oncoming flow is essentially the superposition, at constant pressure P, of a uniform flow with constant velocity U upon a shear flow with a linear velocity distribution. Sakurai[2] extended the problem to the case of an infinite flat plate with uniform suction at the plate. The author[3] solved Sakurai's problem by replacing the viscous liquid by a Maxwellian eslastico-viscous liquid. But Maxwell liquid is not a general relation which explains the flow behaviour of a real elastico-viscous liquid. So in this note we have discussed the same problem with the constitutive equation :

$$p_{ik} + \lambda_1 \left[\frac{Dp_{ik}}{Dt} - p_{ij}\, d_{jk} - p_{jk}\, d_{ij} \right] = 2\,\mu \left[d_{ik} + \lambda_2 \left(\frac{Dd_{ik}}{Dt} - 2 d_{ij}\, d_{jk} \right) \right] \quad (1)$$

where λ_1 and λ_2 are time constants, μ is the static or zero shear rate viscosity. The material time derivative $\frac{D}{Dt}$ is

$$\frac{Da_{ij}}{Dt} \equiv \frac{\partial a_{ij}}{\partial t} + v\, \frac{\partial a_{ij}}{\partial x_k} - w_{ik} a_{kj} + \quad {}_{kj}\, a_{ik},$$

where

$$w_{ij} = \frac{1}{2} \left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] \text{ is the vorticity tensor,}$$

and

$$d_{ij} = \frac{1}{2} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \text{ is the rate of strain tensor.}$$

Here p_{ik} (reduced stress tensor) $= t_{ik}$ (stress tensor) $+ p\delta_{ik}$, where p is a scalar pressure and v_i is the velocity vector. This is known as Oldroyd[4] B liquid.

X-axis is taken along the plate and Y-axis perpendicular to it. For the flow past an infinite flat plate, conditions will depend on y only. Hence, the velocity field can be taken as

$$u = u\,(y); \quad v = v\,(y); \; w = 0 \quad (2)$$

The boundary conditions of the problem are

$$y = 0 : u = 0,\ v = v_o\ ; \quad y \to \infty : u \to U + \Omega\, y \tag{3}$$

The velocity field (2) is compatible with the equation of continuity if

$$\frac{dv}{dy} = 0,$$

which on integration gives

$$v = \text{constant} = v_o, \tag{4}$$

v_o being the constant normal velocity at the plate.

The stress-strain rate relations for the elastico-viscous liquid are reduced to

$$p_{xx} + \lambda_1 \left[v_o \frac{dp_{xx}}{dy} - 2 \frac{du}{dy}\ _{xy} \right] = -2\, \mu\, \lambda_2 \left(\frac{du}{dy} \right)^2, \tag{5}$$

$$p_{xy} + \lambda_1 \left[v_o \frac{dp_{xy}}{dy} - \frac{du}{dy} p_{yy} \right] = \mu \left[\frac{du}{dy} + \lambda_2\, v_o \frac{d^2u}{dy^2} \right], \tag{6}$$

$$p_{yy} + \lambda_1\, v_o \frac{dp_{yy}}{dy} = 0 \tag{7}$$

Equation (7) shows that $p_{yy} = 0$ is a particular solution of this.
Hence, putting $p_{yy} = 0$ in equation (6) we get

$$p_{xy} + \lambda_1\, v_o \frac{dp_{xy}}{dy} = \mu \left[\frac{du}{dy} + \lambda_2\, v_o \frac{d^2u}{dy^2} \right] \tag{8}$$

The momentum equations now reduce to

$$\rho\, v_o \frac{du}{dy} = - \frac{\partial p}{\partial x} + \frac{dp_{xy}}{dy}, \tag{9}$$

$$0 = - \frac{\partial p}{\partial y} \tag{10}$$

Equation (9) shows that $\frac{\partial p}{\partial x}$ is a function of y,

Hence, from equation (10) $\frac{\partial p}{\partial y} = 0.$

$$\frac{\partial p}{\partial x} = \alpha, \quad \text{a constant} \tag{11}$$

which gives the pressure distribution

$$p = \alpha x + \beta$$

where β is a constant.

Eliminating p_{xy} between (8) and (9), we have

$$\mu\, \lambda_2\, v_o \frac{d^3u}{dy^3} + (\mu - \lambda_1 \rho\, v_o{}^2) \frac{d^2u}{dy^2} - \rho\, v_o \frac{du}{dy} - \alpha = 0 \tag{12}$$

The solution of this equation subject to boundary condition $u=0$ at $y=0$ is

$$u = A\,(e^{m_1 y} - 1) + B\,(e^{m_2 y} - 1) - \frac{\alpha}{\rho v_o}\, y, \tag{13}$$

TABLE 1

BOUNDARY LAYER THICKNESS VALUES OF $\frac{\delta v_o}{\nu}$

K_c \ R_c	0·1	0·3	0·5	0·7
0·2	0·909	..	..	..
0·4	0·740	0·923	..	..
0·6	0·588	0·789	0·943	..
0·8	0·435	0·666	0·813	0·946

where A and B are constants of integration and

$$\left.\begin{matrix} m_1 \\ m_2 \end{matrix}\right\} = \frac{v_o}{2\nu}\,\frac{1}{K_c}\left[-(1-R_c) \pm \left\{(1-R_c)^2 + 4K_c\right\}^{\frac{1}{2}}\right] \quad (14)$$

where

$$R_c = \frac{\lambda_1 \rho v_o^2}{\mu} \text{ and } K_c = \frac{\lambda_2 \rho v_o^2}{\mu}$$

Since

$$[(1-R_c)^2 + 4K_c]^{\frac{1}{2}} > (1-R_c),$$

the expression in the square bracket in (14) will have one positive value and the other negative value.

Now we shall study two cases:

(*i*) $v_o < 0$, which corresponds to fluid suction at the plate; and (*ii*) $v_o > 0$, which corresponds to fluid injection at the plate.

Case (I) $v_o < 0$

In this case $m_1 < 0$ and $m_2 > 0$. From the condition at infinity we have

$$B = 0, \quad A = -U \text{ and } \Omega = -\frac{\alpha}{\rho v_o}.$$

Hence

$$u = U + \Omega y - U e^{m_1 y} \quad (15)$$

If $R_c = K_c$, the solution reduces to viscous liquid case.

If δ is the order of boundary layer thickness, $\frac{\delta v_o}{\nu} \sim 2K_c / [\{(1-R_c)^2 + 4K_c\}$ $-(1-R_c)]$ and its value for different values of R_c and K_c is given in Table 1, R_c being greater than K_c by definition of the fluid. Table 1 shows that the boundary layer thickness decreases with the increase in relaxation time, but increases with the retardation time.

The skin-friction at the plate τ_o is

$$\tau_o = -\left[\frac{\alpha\mu}{\rho v_o} + \rho\, v_o\, U \right] \tag{16}$$

which shows that the shear stress at the wall is not affected by the elasticity of the fluid but it is not the same as in the ordinary hydrodynamic flow in the absence of vorticity. In shear flow the skin-friction increases.

Case (II) $v_o > 0$

In this case, $m_1 > 0$ and $m_2 < 0$. From the condition at infinity, we have

$$A = 0,\ B = -U \text{ and } \Omega = -\frac{\alpha}{\rho v_o}$$

Hence from (13)

$$u = U + \Omega y - U e^{m_2 y} \tag{17}$$

If $R_c = K_c = 0$, which corresponds to viscous case, we obtain m_2 to be infinity which proves that in viscous case a solution is not possible if there is fluid injection at the plate.

The skin-friction at the plate τ_o in this case is also as given in (16). But since there is fluid injection at the plate, $v_o > 0$ and τ_o becomes negative which is clearly impossible. Hence, the only possible case for maintaining a laminar motion is to suck out fluid at the plate.

If there is no vorticity in the oncoming flow, we can put $\alpha = 0$ in our equations and the flow over a flat plate in the absence of a pressure gradient comes as particular case.

If $v_o = 0$, *i.e.*, no suction is assumed equation (12), subject to the boundary conditions, (3) its has only the plane Poiseculle flow as solution

$$u = \frac{\alpha}{2\mu} y^2 + \beta\, y, \qquad \beta = \text{constant}$$

It is, therefore, interesting to note that the velocity field is affected by the elasticity of the fluid only if the suction is present.

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