A SAMPLING INSPECTION SCHEME BASED ON SIMPLE MARKOV CHAINS

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(*Received 29 June 1964*)

Using the principles of Markov Chains, an inspection procedure for classifying given number of lots, each consisting of a given number of items, into three categories, say A, B and C has been developed. The average percentage of defectives in the accepted lots of different categories have been worked out for a given set of parameters.

For classifying lots consisting of certain number of produced items into three categories superior, ordinary and inferior or type A, type B and type C, a sampling inspection scheme based on Markov Chains has been used. Such a classification may be necessary from the point of view of usage of items under various circumstances or for the industries in fixing different prices for different type of items.

We assume here that the production process is continuous and under statistical control. Further, it is possible to divide the produced items into lots of fixed number of items (which is determined by considerations of despatching or consumers requirements). Moreover, the inspection is carried out by attributes *i.e.*, either an item is declared defective or non-defective.

SCHEME

From the first lot we take a sample and note the number of defective items in it. On this basis we declare the lot to be of type A, type B, or type C, according to some criteria fixed in advance.

Now the size of the sample to be taken from the second lot will depend upon the type in which the first lot has been classified. The sample size would be n_1, n_2 or n_3 according as the first lot was classified as type A, type B, or type C respectively. The same procedure is followed for all the subsequent lots. In general, sample size from kth lot would be n_1, n_2, or n_3 according as the $(k-1)$th lot was classified as type A, type B or type C respectively; $k=2, 3, \ldots r$ (say).

MATHEMATICAL DERIVATIONS

Let P_{ij} be the probability of a lot being classified as type j ($j=1, 2, 3$ *i.e.* A, B, C) when the size of the sample from the lot is n_i ($i=1, 2, 3$) and let $\overset{(k)}{P_j}$ denote the probability of classifying the kth lot as type j.
Then evidently:

$$\overset{(k)}{P_j} = \sum_{i=1}^{3} p_{ij} \overset{(k-1)}{P_i} \qquad j = 1, 2, 3; \quad k = 2, 3, \ldots r$$

For further discussion we relate the above sampling inspection scheme to a Markov chain. A lot is said to be in state j according as it is classified as type j ($j = 1, 2, 3$). The transition probability matrix is

$$[\, p_{ij} \,]$$

Obviously,

$$\sum_{j=1}^{3} p_{ij} = 1 \qquad i = 1, 2, 3; \quad j = 1, 2, 3$$

Let the steady state probability for any lot to be classified as type j ($j = 1, 2, 3$) be denoted by P_j so that [1]

$$P_j = \frac{\Delta_{jj}}{\sum_{j=1}^{3} \Delta_{jj}} \qquad j = 1, 2, 3$$

where Δ_{jj} is the co-factor of $(1-p_{jj})$ in the determinant

$$| I - p_{ij} |$$

where I is the unit matrix and

$$\Delta_{jj} = (1-p_{ii})(1-p_{kk}) - p_{ik}\, p_{ki} \qquad i, j, k = 1, 2, 3. \quad i \neq j \neq k$$

Then the probability that kth lot is in state i, given that 1st lot was in state j, is given as [1,2]

$$\overset{k}{P_i}(j) = P_i + a_{1i}\, \lambda_1^{k-2} + a_{2i}\, \lambda_2^{k-2} \qquad k = 2, 3, \ldots r$$

where 1, λ_1 and λ_2 are the characteristic roots of the matrix $[P_{ij}]$ and a_{1i} and a_{2i} are constants.

Calculations show that

$$\lambda_1 = \left[\sum \left(p_{ii} - 1 \right) + \left\{ \left(1 - \sum p_{ii} \right)^2 - 4 \,| p_{ij} | \right\}^{\frac{1}{2}} \right] \div 2$$

and

$$\lambda_2 = \left[\left(\sum p_{ii} - 1 \right) - \left\{ \left(1 - \sum p_{ii} \right)^2 - 4 \,| p_{ij} | \right\}^{\frac{1}{2}} \right] \div 2$$

where $| p_{ij} |$ denotes the determinant of the matrix $[p_{ij}]$. Further, using the initial conditions we get the equations:

$$p_{ji} = P_i + a_{1i} + a_{2i}$$

$$p_{ji} = P_i + a_{1i}\lambda_1 + a_{2i}\lambda_2$$

which on solving for a_{1i} and a_{2i} give

$$a_{1i} = \left\{ (p_{ji} - P_i) - \lambda_2 (p_{ji} - P_i) \right\} \div (\lambda_1 - \lambda_2)$$

and

$$a_{2i} = \left\{ (p_{ji} - P_i) - (p_{ji} - P_i) \right\} \div (\lambda_1 - \lambda_2)$$

where

$$p_{ji} = \sum_{k=1}^{3} p_{jk}\, p_{ki}$$

It can be easily seen that for larger r,

$$\sum_{k=2}^{r} \overset{k}{P_i}(j) = (r-1) P_i + \frac{a_{1i}}{1-\lambda_1} + \frac{a_{2i}}{1-\lambda_2}$$

Now first of all, we proceed to derive the expression for total amount of inspection. To this end we see that expected size of the sample from the kth lot, given that first lot was classified as type j, is

$$\sum_{i=1}^{3} n_i \overset{k-1}{P_i} (j)$$

Hence total number of items inspected from all the r lots, given that first lot was classified as type j is

$$n + n_j + \sum_{i=1}^{3} \sum_{k=3}^{r} n_i \overset{k-1}{P_i} (j)$$

$$= n + n_j + (r-2) \sum_{i=1}^{3} n_i P_i + \sum_{i=1}^{3} \left(\frac{a_{1i}}{1-\lambda_1} + \frac{a_{2i}}{1-\lambda_2} \right) n_i$$

where n is the sample size from the first lot.

The number of lots classified as type j for large r is independent of the classification of the first lot and is approximately[1]

$$rP_j \qquad (j = 1, 2, 3)$$

To derive the expression for expected percentage of defective items in the lots classified as type A, type B and type C we use the following criteria for classifying a lot as type A, type B or type C—when the sample size from a lot is n_i, the lot is classified as type A if the number of defectives in the sample is $\leqslant u_i$, as type B if the number of defective is $> u_i$ but $\leqslant v_i$ and as type C otherwise.

First we shall derive the expression for expected number of defectives in lots classified as type A.

Let us consider the kth lot. This lot can be classified as type A; either

(1) with the help of a sample of size n_1 from it,

or (2) with the help of a sample of size n_2 from it,

or (3) with the help of a sample of size n_3 from it.

Now, probability of (1) is

(Probability that $(k-1)$th lot is classified as type A)
$\times$(Probability that kth lot is classified as type A)

$$= \overset{k-1}{P_1} (l) \times \sum_{j=0}^{u_1} {}^{n_i}C_j \, p^j \, q^{n_1-j}$$

(assuming that first lot was classified as type l; $l = 1, 2,$ or 3 and the process fraction defective is p).

Similarly the probabilities of (2) and (3) are

$$\overset{k-1}{P_2} (l) \times \sum_{j=0}^{u_2} {}^{n_2}C_j \, p^j \, q^{n_2-j}$$

and

$$\overset{k-1}{P_3} (l) \times \sum_{j=0}^{u_3} {}^{n_3}C_j \, p^j \, q^{n_3-j}$$

respectively.

Thus expected number of defectives in kth lot classified as type A

$$= \sum_{i=1}^{3} \text{(Probability that it is classified as type A with the help of a sample of size } n_i\text{)} \times \text{(Expected number of defectives in a sample of size } n_i \text{ given that it has been classified as type A)}$$

$$= \sum_{i=1}^{3} P_i^{k-1}(l) \times \sum_{j=0}^{u_i} j\ {}^{n_i}C_j\, p^j\, q^{n_i-j} \times \frac{N-n_i}{n_i}$$

Summing over all the lots we get

Expected number of defectives in lots classified as type A

$$= \sum_{i=1}^{3} \frac{N-n_i}{n_i} \left(\sum_{j=0}^{u_i} j\ {}^{n_i}C_j\ p^j\ q^{n_i-j} \right) \left(\sum_{k=2}^{r} P_i^{k-1}\ (l) \right)$$

$$= \sum_{i=1}^{3} \frac{N-n_i}{n_i} \left(\sum_{j=0}^{u_i} j\ {}^{n_i}C_j\ p^j\ q^{n_i-j} \right) \left(\sum_{k=3}^{r} P_i^{k-1}\ (l) \right)$$

$$+3.\frac{N-n_l}{n_l} \sum_{j=0}^{u} {}^{n_l}C_j\ p^j\ q^{n_l-j}$$

$$= \sum_{i=1}^{3} \frac{N-n_i}{n_i} \left(\sum_{j=0}^{u_i} j\ {}^{n_i}C_j\ p^j\ q^{n_i-j} \right) \left(\sum_{k=2}^{r-1} P_i^{k}\ (l) \right)$$

$$+3.\ \frac{N-n_l}{n_l} \sum_{j=0} {}^{n_l}C_j\ p^j\ q^{n_l-j}$$

$$= (r-1) \sum_{i=1}^{3} \frac{N-n_i}{n_i} P_i \cdot \left(\sum_{j=0}^{u_i} j\ {}^{n_i}C_j\ p^j\ q^{n_i-j} \right)$$

$$+3.\frac{N-n_l}{n_l} \sum_{j=0}^{u_l} j\ {}^{n_l}C_j\ p^j\ q^{n_l-j}$$

$$+ \sum_{i=1} \left(\frac{a_{1i}}{1-\lambda_1} + \frac{a_{2i}}{1-\lambda_2} \right) \cdot \frac{N-n_i}{n} \left(\sum_{j=0}^{u_i} j\ {}^{ni}C_j\ p^j\ q^{n_i-j} \right)$$

Similarly expected number of defectives in lots classified as type B is

$$(r-1) \sum_{i=1}^{3} \frac{N-n_i}{n_i} P_i \left(\sum_{j=u_i+1}^{v_i} j\ {}^{n_i}C_j\ p^j\ q^{n_i-j} \right) +3.\ \frac{N-n_l}{n_l}$$

TABLE 1

NUMBER OF LOTS CLASSIFIED IN VARIOUS CATEGORIES AND PERCENTAGE OF DEFECTIVES IN LOTS OF EACH TYPE

($r=101$, $N=1000$, $n_1=50$, $n_2=75$, $n_3=100$)

Type of lots	Process average fraction of defective etc.		
	p=10%	p=5%	p=1%
	$u_1=2$; $u_2=3$; $u_3=4$ $v_1=6$; $v_2=9$; $v_3=12$	$u_1=2$; $u_2=3$; $u_3=4$ $v_1=6$; $v_2=9$; $v_3=12$	$u_1=0$; $u_2=1$; $u_3=1$ $v_1=2$; $v_2=3$; $v_3=4$
A	5·5 lots 3·00% defectives	51·1 lots 2·75% defectives	67·6 lots 0·22% defectives
B	72·8 lots 9·05% defectives	48·3 lots 7·10% defectives	31·2 lots 2·46% defectives
C	21·7 lots 14·33% defectives	0·6 lots 10·3% defectives	1·2 lots 6·0% defectives

N.B.—1. For computation, we have assumed that the first lot is classified as type A, type B or type C according to process average fraction defective *e.g.*, when $p=10\%$, we have assumed that the first lot was classified as type B because, approximately the lots containing percentage defective between 4% and 12% would be classified as type B.

2. Parameters u_i, v_i are defined as: When the sample size is n_i, the lot is classified as type A if the number of defectives in the sample is $\leq u_i$, type B if the number of defectives in the sample is $> u_i$ but $\leq v_i$ and type C otherwise.

$$\sum_{j=0}^{v_l} j \; {}^{n_l}C_j \; p^j q^{n_l-j} + \sum_{i=1}^{3} \frac{N-n_i}{n_i} \left(\frac{a_{1i}}{1-\lambda_1} + \frac{a_{2i}}{1-\lambda_2}\right) \left(\sum_{j=u_i+1}^{v_i} j \; {}^{n_i}C_j \; p^j q^{n_i-j}\right)$$

and expected number of defectives in lots classified as type C is

$$(r-1) \sum_{i=1}^{3} \frac{N-n_i}{n_i} P_i \left(\sum_{j=v_i+1}^{n_i} j \; {}^{n_i}C_j \; p^j \; q^{n_i-j}\right)$$

$$+ \frac{N-n_l}{n_l} \sum_{j=0}^{u_l} j \; {}^{n_l}C_j \; p^j \; q^{n_i-j}$$

$$+ \sum_{i=1}^{3} \frac{N-n_i}{n_i} \left(\frac{a_{1i}}{1-\lambda_1} + \frac{a_{2i}}{1-\lambda_2}\right) \left(\sum_{j=v_i+1}^{n_i} j \; {}^{n_i}C_j \; p^j \; q^{n^i-j}\right)$$

COMPARISON WITH SIMILAR SCHEME WHEN THE SAMPLE SIZE IS CONSTANT

The scheme presented here is quite flexible as regards size of samples to be taken from various lots for inspection. When the quality of the produced items remains consistently good the amount of inspection will be minimum for the scheme and when the quality of the produced items remains consistently bad the amount of inspection will be maximum for the scheme. This flexibility is not achieved by a scheme if the sample size is kept constant.

It has not been possible to discuss the choice of various parameters involved; firstly because they are too many in number; secondly because the expressions for the expected number of defectives in lots of a particular type etc. are too complicated. However, Table 1 gives a general idea about the number of lots and percentage of defective items in these lots for a certain set of values of p, u_i, v_i $(i=1, 2, 3)$ while r, N and n_i $(i=1, 2, 3)$ have been fixed.

REFERENCES

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