A NOTE ON QUANTITATIVE TREATMENTS IN A GENERAL INCOMPLETE BLOCK DESIGN

A. M. KSHIRSAGAR

Defence Science Laboratory, Delhi

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In some cases, the treatments in a design are formed by adding varying amounts of an "additive" factor to a "basic" factor. This leads to a relationship between the effects of treatments and the amounts of the additives. The parameters of interest are, therefore, the coefficients in this response curve rather than the usual treatment effects. The problem of inference about these new parameters is considered in this note.

John[1] has described a balanced incomplete block design (B.I.B.) in which the nine treatments, that were used, were quantitative, rather than qualitative being actually two additives each at 4 levels and a third one at only one level. In such situations, the treatment effects alone are not of interest but one should consider the relationship of response between the treatment and the amount of additive used. The parameters of interest are thus the coefficients in the response curve and their significance can be tested by sub-dividing the adjusted treatment sum of squares into meaningful components. John has done this only for a B.I.B. design, in the case of 2 basic factors and additives at 4 levels each. Here we consider a general incomplete block design wherein there are m basic factors to each of which an additive at different number of levels is added. The analysis of such a general design is derived.

Notation and set-up—Let us suppose that $0, 1, 2, \ldots\ldots\ldots l_x$ -1 parts of an additive are added to the xth basic factor ($x = 1, 2, \ldots\ldots m$). Therefore the number of treatments tested in the general incomplete block design are:

$$\sum_{x=1}^{m} l_x$$

If besides these, an additional control treatment is also used, the total number of treatments will be

$$v = \sum_{x=1}^{m} l_x + 1$$

Let the treatments excluding control be serially numbered from 1 to (v—1) in such a way that the ith treatment is obtained by adding y parts of the additive to the xth basic factor.

If (x=1,2......m; y=0, 1, ...l_x —1), then

$$i = (l_1 + \ldots\ldots + l_{x-1}) + y + 1 \tag{2·1}$$

Let t_i denote the effect of the ith treatment. We can describe the relationship between t_i and y, the amount of the additive used, by the response curve

$$t_i = b_{x,o} P_o (y) + b_{x,1} P_1 (y) + \ldots\ldots + b_{x,\, l_x-1} P_{l_x-1}(y) \tag{2·2}$$

where $P_r\ (y)$ is the rth degree orthogonal polynomial in y. We now wish to estimate the coefficients $b_{x,y}$ in (2·2) and test their significance. We shall assume that the design is a connected one so that all the treatment contrasts are estimable.

Let t denote the column vector of all the treatments, the last element t_v being the effect of the control. Let further t_x be the column vector of the l_x elements t_i , where i is given by (2·1) and y runs from 0 to $l_x - 1$. Then

$$t' = [\,t_1' \mid \ldots\ldots \mid t'_x \mid \ldots\ldots \mid t'_m\,] \tag{2·3}$$

Let

$$b'_x = [\,b_{x,o}\ b_{x,1},\ \ldots\ldots,\ b_{x,\,l_x-1}\,]$$

Analysis : Let us assume that the orthogonal polynomials $P_r\ (y)$ are so normalized that

$$\sum_y P_r^{\,2}(y) = 1 \tag{3·1}$$

Then (2·2) can be written as

$$t_x = P_x\, b_x \quad (x = 1, \ldots\ldots\ m) \tag{3·2}$$

where P_x is the $l_x \times l_x$ matrix whose columns are $P_0(y), P_1(y), \ldots\ldots\ldots\ldots P_{l_x-1}(y)$ and the rows correspond to $y = 0, 1, \ldots\ldots\ldots\ l_x - 1$. On account of (3·1), P_x is an orthogonal matrix and hence from (3·2)

$$b_x = P_x' t_x \tag{3·3}$$

This shows that

(3·4) $\sqrt{l_x}\ b_{x,o} =$ Sum of all the elements of t_x while $b_{x,1}, \ldots\ldots\ b_{x,\,l_x-1}$ are all contrasts in the elements of t_x . Thus $b_{x,o}$ is not estimable but $b_{x,y}\ (y \neq 0)$ are all estimable. The average effect of the xth basic factor (without any additive) is measured by $b_{x,o}$.

It is well-known in the theory of design of experiments (see for example, *Chakrabarti*[2]) that the best estimate of any treatment contrast $h't$ is obtained by putting $\hat{t}$ for **t** where $\hat{t}$ is any solution of the reduced normal equantions

$$Q = Ct \tag{3·5}$$

where Q is the vector of the adjusted treatment totals and $\sigma^2 C$ is the variance-covariance matrix of Q and is obtainable from the incidence matrix of the design. The variances and covariances of these best estimates of treatment contrasts can be easily obtained by expressing them in terms of Q using (3·5) and

$$V(Q) = C\sigma^2 \tag{3·6}$$

$$\sigma^2 = \text{variance of the yield of a plot.}$$

The best estimates of the parameters $b_{x,y}$ ($x = 1, \ldots\ldots\ldots, m;\ y \neq 0,\ y = 1, \ldots\ldots\ldots, l_x - 1$) can, therefore, be easily obtained from the above considerations. The sum of squares for testing the significance of any $b_{x,y}\ (y \neq 0)$ is

$$\frac{\sigma^2(\text{best estimate of } b_{x,y})^2}{\text{variance of } b_{x,y}} \tag{3·7}$$

and has 1 d.f. It should be noted that the estimates of $b_{x,y}$ are correlated, in general, and their sums of squares are, therefore, not additive. For a B.I.B. design, however, C has a special, structure and, therefore, the various estimates are uncorrelated, as noted by John.

We now derive the sum of squares for testing the significance of the differences among the m basic factors themselves. These differences are measured by the differences among $b_{x,o}(x = 1, \ldots\ldots, m)$; $b_{x,o}$ is not estimable but a contrast among $b_{x,o}/\sqrt{l_x}$ is estimable. Choose any $m-1$ linearly independent contrasts among $b_{x,o}/\sqrt{l_x}$ $(x = 1, \ldots\ldots\ldots m)$. Let U denote the column vector of the best estimates of these $m-1$ contrasts. The variance-covariance matrix V of U can be obtained by the procedure outlined above. The required s.s. is then $\frac{1}{\sigma^2} U'V^{-1}U$ with $m-1$ d.f.

Lastly, we consider the difference between the m basic factors and the control. It is measured by

$$\frac{1}{v-1}\sum_{x=1}^{m} \sqrt{l_x}\, b_{x,o} - t_v$$

$$= \frac{1}{v-1}\sum_{i=1}^{v-1} t_i - t_v$$

Its estimates, variance and sum of squares can then the easily obtained as before.

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