OPTIMUM STAGING WITH VARYING THRUST ATTITUDE ANGLE

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Optimum staging programme for step rockets of arbitrary number of stages having different specific impulses and mass fractions with stages is derived, the optimization criterion being minimum take-off weight for a desired burntout velocity at an assigned altitude. Variation of thrust attitude angle from stage to stage and effects of gravity factor are taken into account. Analysis is performed for a degenerate problem obtained by relaxing the altitude constraint and it has been shown that problems of Weisbord, Subotowicz, Hall & Zambelli and Malina & Summerfield are the particular cases of the degenerate problem.

NOMENCLATURE

W_T = initial gross weight of the rocket

W_{PL} = payload weight

ψ_k = average thrust attitude angle with horizontal of the kth stage

I_n = specific impulse of the nth stage

λ_n = gross weight payload ratio of the nth stage

t_n = burning time of the nth stage

μ_n = mass fraction of the nth stage *i.e.* the ratio between the weight of the propellant and weight of the nth stage.

r_{on} = initial thrust to initial gross weight ratio of the nth stage

g = acceleration due to gravity

N = total number of stages

K_n, β'_n = structural factor for the nth stage

C_n, C'_n, w'_n = propellant exhaust velocity of the nth stage

μ'_n, r'_n = ideal mass parameter of the nth stage

ξ'_n = propellant parameter of the nth stage

λ'_n = payload parameter of the nth stage

θ'_n = construction parameter of the nth stage

A' = total payload parameter

Subscript

(C) = denotes constant values

Superscript

$(')$ = denotes corresponding notation of ref. 2 and 3.

In the performance of step rockets, optimum staging plays dominant role from two considerations, either to derive maximum performance with preassigned resources or to achieve a required performance with the minimum resources. The optimization of rocket staging for minimum gross weight of the rocket has been considered by a number of authors[1–7]. Malina & Summerfield[4] considered optimization of homogeneous stages for a minimum gross weight to achieve a given burntout velocity. Relaxing the restrictions of constant specific impulse and structural factor in Malina & Summerfield problem, Goldsmith[5] offered a solution for two stage rockets when the structural weights were proportional to the fuel weight and the powerplant weights were proportional to the stage gross weight. Weishbord[1], Subotowicz[2] and Hall & Zambelli[3] have presented the general solution for minimum gross weight for non-homogeneous stages holding good for arbitrary number of stages. In all the above investigations, the variation of thurst attitude angles with stages is neglected and a field-free space is assumed.

This paper presents optimum staging for minimum gross weight of arbitrary number of stages to achieve a given burntout velocity at an assigned altitude measured vertically from the point of location of the rocket firing. Variation of thrust attitude angle with stages and effect of gravity are taken into consideration. The results are general in so far as specific impulses and mass fractions may have different values with stages. Then altitude constraint is relaxed and the problem reduces to the optimization problem where variations of thrust attitude angle with stages and gravity effect are included and velocity equation is the sole restrictive condition. The results of earlier investigations[1–3] are deduced as particular cases of this velocity constraint optimization problems.

OPTIMIZATION ANALYSIS

The relationship between W_T, W_{PL} and λ can be written as

$$\frac{W_T}{W_{PL}} = (\lambda_1)\ (\lambda_2)\ \ldots\ldots\ldots\ (\lambda_N) \tag{1}$$

A first integral of the equation of motion for N-stage rocket gives burntout velocity V (see appendix) as

$$V = \sum_{n=1}^{N-1} V_n \prod_{k=n}^{N-1} \cos \xi_k + V_N \tag{2}$$

where

$$\xi_k = \psi_{k+1} - \psi_k \qquad [k = 1, 2, \ldots\ldots\ldots, (N-1)]$$

and

$$V_n = g\, I_n \log \left[\frac{\lambda_n}{\lambda_n (1-\mu_n) + \mu_n} \right] - g t_n \sin \psi_n$$

Burntout altitude H is given by the second integral (see appendix) where

$$H = \sum_{n=1}^{N} \left(V_1 \prod_{k=1}^{n-1} \cos \xi_k + V_2 \prod_{k=2}^{n-1} \cos \xi_k + V_3 \prod_{k=3}^{n-1} \cos \xi_k + \ldots + V_{n-1} \cos \xi_{n-1} \right) \times \sin \psi_n t_n$$

$$+\sum_{n=1}^{N} (g t_n I_n \sin \psi_n) \left[1 - \frac{\log \dfrac{\lambda_n}{\lambda_n (1-\mu_n)+\mu_n}}{\dfrac{(\lambda_n - 1)\mu_n}{\lambda_n(1-\mu_n)+\mu_n}} \right] - \sum_{n=1}^{N} \frac{g \sin^2 \psi_n t^2_n}{2} \quad (3)$$

where

$$t_n = \frac{\mu_n(\lambda_n - 1) I_n}{(\lambda_n) r_{on}}$$

Let the restrictive conditions for payload ratios λ be given by

$$V = V_c \quad (4)$$

$$H = H_c \quad (5)$$

If α and β be the Lagrangian undetermined multipliers, the optimization equations can be written as

$$\frac{\partial}{\partial \lambda_n}\left(\frac{W_T}{W_{PL}}\right) + \alpha \frac{\partial V}{\partial \lambda_n} + \beta \frac{\partial H}{\partial \lambda_n} = 0 \quad (n = 1, 2, \ldots\ldots\ldots\ldots, N) \quad (6)$$

Differentiating (1), (2) and (3) partially with respect to λ_n and substituting in (6) optimization equations yield

$$\frac{\mu_n I_n \sin\psi_n}{r_{on}(\lambda_n)} \Bigg[\left(\frac{r_{on}\lambda_n}{\sin\psi_n \lambda_n (1-\mu_n)+\mu_n} - 1 \right) \Big\{ \alpha g + \beta g \Big(\sin\psi_{n+1} t_{n+1} \cos\xi_n$$

$$+ \sin\psi_{n+2} t_{n+2} \prod_{k=n}^{n-1} \cos\xi_k + \sin\psi_{n+3} t_{n+3} \prod_{k=n}^{n+2} \cos\xi_k + \ldots + \sin\psi_N t_N \prod_{k=n}^{N-1} \cos\xi_k \Big) \Big\}$$

$$+ \beta \left(V_1 \prod_{k=1}^{n-1} \cos\xi_k + V_2 \prod_{k=2}^{n-1} \cos\xi_k + \ldots\ldots + V_{n-1}\cos\xi_{n-1} + V_n \right) \Bigg] = -\frac{W_T}{W_{PL}}$$

$$(n = 1, 2, \ldots\ldots\ldots\ldots\ldots, N) \quad (7)$$

Equations of the type (7) together with (4) and (5) form $(N + 2)$ nonlinear algebraic equations. For preassigned values of specific impulses, mass fractions, initial thrust to weight ratios and average thrust attitude angles, the above equations can be solved by iterative processes giving optimum values, which on substitution in (1) will give minimum gross take-off weight of the rocket.

Reduction of optimization equations

The number of nonlinear algebraic equations giving optimum payload ratios can be reduced to N only by elimination of Langrangian multipliers. Due to (1), (6) may be written as

$$\alpha\lambda_n \frac{\partial V}{\partial \lambda_n} + \beta\lambda_n \frac{\partial H}{\partial \lambda_n} = -\frac{W_T}{W_{PL}}; \; n = (1, 2, \ldots\ldots\ldots\ldots, N) \quad (8)$$

or

$$\left[(\lambda_n) \frac{\partial V}{\partial \lambda_n} - (\lambda_{n+1}) \frac{\partial V}{\partial \lambda_{n+1}} \right] \left[(\lambda_{n+1}) \frac{\partial H}{\partial \lambda_{n+1}} - (\lambda_{n+2}) \frac{\partial H}{\partial \lambda_{n+2}} \right]$$

$$= \left[(\lambda_{n+1}) \frac{\partial V}{\partial \lambda_{n+1}} - (\lambda_{n+2}) \frac{\partial V}{\partial \lambda_{n+2}} \right] \left[(\lambda_n) \frac{\partial H}{\partial \lambda_n} - (\lambda_{n+1}) \frac{\partial H}{\partial \lambda_{n+1}} \right]$$

$$[\, n = 1, 2, \ldots\ldots\ldots\ldots, (N - 2) \,] \quad (9)$$

(9) together with (4) and (5) will give the optimum values of λ. Now fr m (2) and (3) after differentiating partially with respect to λ_n, we obtain

$$\frac{\partial V}{\partial \lambda_n} = \frac{\mu_n g I_n \sin \psi_n}{r_{on} (\lambda_n)^2} \left[\frac{r_{on} \lambda_n}{\sin \psi_n \{ \lambda_n (1 - \mu_n) + \mu_n \}} - 1 \right] \tag{10}$$

and

$$\frac{\partial H}{\partial \lambda_n} = \frac{\mu_n I_n \sin \psi_n}{r_{on} (\lambda_n)^2} \Bigg[g \Bigg(\sin \psi_{n+1} t_{n+1} \cos \xi_n + \sin \psi + \prod_{k=n}^{n+1} \cos \xi_k + \dots + \sin \psi_N t_N \prod_{k=n}^{N-1} \cos \xi_k \Bigg) + \Bigg(V_1 \prod_{k=1}^{n-1} \cos \xi_k + V_2 \prod_{k=2}^{n-1} \cos \xi_k + \dots\dots V_{n-1} \cos \xi_{n-1} + V_n \Bigg) \Bigg] \tag{11}$$

Substitution of (10) and (11) in (9) reduces (9) in terms of the known rocket parameters and unknown payload ratios.

Application to a four stage rocket

Let $\lambda_n \dfrac{\partial V}{\partial \lambda_n} = \phi_n$ and $\lambda_n \dfrac{\partial H}{\partial \lambda_n} = \bar{\theta}_n$

(9) for a four stage rocket will become

$$(\phi_1 - \phi_2)(\bar{\theta}_2 - \bar{\theta}_3) = (\phi_2 - \phi_3)(\bar{\theta}_1 - \bar{\theta}_2) \tag{12}$$

and

$$(\phi_2 - \phi_3)(\bar{\theta}_3 - \bar{\theta}_4) = (\phi_3 - \phi_4)(\bar{\theta}_2 - \bar{\theta}_3) \tag{13}$$

where

$$\phi_1 = \frac{\mu_1 g I_1 \sin \psi_1}{r_{o1} \lambda_1} \left[\frac{r_{01} \lambda_1}{\sin \psi_1 \{ \lambda_1 (1-\mu_1) + \mu_1 \}} - 1 \right]$$

$$\bar{\theta}_1 = \frac{\mu_1 g I_1 \sin \psi_1}{r_{o1} \lambda_1} \left[g \left(\sin \psi_2 t_2 \cos \xi_1 + \sin \psi_3 t_3 \cos \xi_1 \cos \xi_2 + \sin \psi_4 t_4 \cos \xi_1 \cos \xi_2 \cos \xi_3 \right) + V_1 \right]$$

$$\phi_2 = \frac{\mu_2 g I_2 \sin \psi_2}{r_{o2} \lambda_2} \left[\frac{r_{o2} \lambda_2}{\sin \psi_2 \{ \lambda_2 (1-\mu_2) + \mu_2 \}} - 1 \right]$$

$$\bar{\theta}_2 = \frac{\mu_2 g I_2 \sin \psi_2}{r_{o2} \lambda_2} \left[g \left(\sin \psi_3 t_3 \cos \xi_2 + \sin \psi_4 t_4 \cos \xi_2 \cos \xi_3 \right) + V_1 \cos \xi_1 + V_2 \right]$$

$$\phi_3 = \frac{\mu_3 g I_3 \sin \psi_3}{r_{o3} \lambda_3} \left[\frac{r_{o3} \lambda_3}{\sin \psi_3 \{ \lambda_3 (1-\mu_3) + \mu_3 \}} - 1 \right]$$

$$\bar{\theta}_3 = \frac{\mu_3 g I_3 \sin \psi_3}{r_{o3} \lambda_3} \left[g \sin \psi_4 t_4 \cos \xi_3 + V_1 \cos \xi_1 \cos \xi_2 + V_2 \cos \xi_2 + V_3 \right]$$

$$\phi_4 = \frac{\mu_4 g I_4 \sin \psi_4}{r_{o4} \lambda_4} \left[\frac{r_{o4}}{\sin \psi_4} \frac{\lambda_4}{\{ \lambda_4 (1 - \mu_4) + \mu_4 \}} - 1 \right]$$

$$\bar{\theta}_4 = \frac{\mu_4 g I_4 \sin \psi_4}{r_{o4} \lambda_4} \left[V_1 \cos \xi_1 \cos \xi_2 \cos \xi_3 + V_2 \cos \xi_2 \cos \xi_3 + V_3 \cos \xi_3 + V_4 \right]$$

The restrictive conditions are

$$V = V_c \tag{14}$$

$$H = H_c \tag{15}$$

Equations (12) to (15) with given rocket parameters constitute four equations in four unknowns λ_1, λ_2, λ_3 and λ_4 and on solution yield optimum values of λ.

Case when β vanishes

In this case condition (5) is dropped and (6) becomes

$$\frac{\partial}{\partial \lambda_n}\left(\frac{W_T}{W_{PL}}\right) + \alpha \frac{\partial V}{\partial \lambda_n} = 0 \tag{16}$$

with the restrictive condition as in (14).

The problems of the references 1—3 may be derived as particular cases of the above problem when $\beta = 0$ and can be solved with the help of (14) and (16).

WEISBORD'S PROBLEM

Weisbord's problem is to find out stage weight distribution for minimum gross weight of N-Staged missiles to achieve performance determined by given final burntout velocity. The problem neglects gravity effects and variation of thrust attitude angles with stages. In ref. 1 miscellaneous hardware consisting of tanks, structure guidance, engines, etc. are lumped together and defined by structural factor K thus :

miscellaneous weight $= K(Mg$—Payload)

Under these assumptions (14) and (16) take the simple form where now V in (14) and (16) is given by

$$V = \sum_{n=1}^{N} V_n \tag{17}$$

where

$$V_n = C_n \log \left[\frac{\lambda_n}{\lambda_n(1-\mu_n) + \mu_n} \right]$$

which is transformed to

$$V_n = C_n \log \left[\frac{1}{K_n + \left(\frac{1-K_n}{\lambda_n}\right)} \right]$$

since

$$\mu_n + K_n = 1$$

substitution of (17) in (16) yields

$$\frac{C_n(1-K_n)}{[\lambda_n K_n + (1-K_n)]} = -\frac{1}{\alpha}\left(\frac{W_T}{W_{PL}}\right) = \text{const.} \tag{18}$$

and the restrictive condition is then

$$\sum_{n=1}^{N} C_n \log\left(\frac{1}{K_n + (1-K_n)\frac{1}{\lambda_n}}\right) = V_c \tag{19}$$

From (18) it follows that

$$\frac{1}{\lambda_n} = \frac{C_{n-1} K_n (1 - K_{n-1})}{(1-K_{n-1})(1-K_n)(C_n - C_{n-1}) + C_n K_{n-1}(1-K \quad)\lambda_{n-1}}$$

$$(n = 2, 3, \ldots\ldots\ldots, N) \qquad (20)$$

(20) and (19) provide the solution of Weisbord problem for the general case of N-stage missile. Taking $N = 3$, one gets from (20) two equations

$$\left.\begin{aligned} \frac{1}{\lambda_2} &= \frac{C_1 K_2 (1 - K_1)}{(1-K_1)(1-K_2)(C_2 - C_1) + C_2 K_1 (1 - K_2)\lambda_1} \\ \text{and} \quad \frac{1}{\lambda_3} &= \frac{C_2 K_3 (1 - K_2)}{(1-K_2)(1-K_3)(C_3 - C_2) + C_3 K_2 (1 - K_3)\lambda_2} \end{aligned}\right\} \qquad (21)$$

obtained by Weisbord as solution for three stage rocket.

SUBOTOWICZ PROBLEM

Subotowicz problem of the optimization of the N-step rocket with different construction parameters and propellant specific impulses in each case can be easily derived from (14) and (16). Nomenclature of ref. 2 are used with a superscript dash here in order to avoid confusion with the notation of the present paper. In ref. 2 the following general relations exist for the rocket parameters:

$$r' = \frac{1}{1-\xi'} \; ; \; \xi' = 1 - (\lambda' + \theta') \; ; \; r' = \frac{1}{\lambda' + \theta'}$$

Since it can be easily varified that

$$\xi'_n = \mu_n \left(1 - \frac{1}{\lambda_n}\right) \qquad (22)$$

and

$$\lambda_n = \frac{1}{\lambda'_n} = (1 - \xi'_n - \theta'_n)^{-1} \qquad (23)$$

Therefore by (22), V_n in (14) and (16) will take the form

$$\begin{aligned} V_n &= I_n \, g \log \left[\frac{\lambda_n}{\lambda_n (1 - \mu_n) + \mu_n}\right] \\ &= w'_n \log \left[\frac{1}{1 - \xi'_n}\right] \end{aligned} \qquad (24)$$

Substitution of (23) and (24) in (16), differentiation, and simplification gives

$$\frac{1}{(1 - \xi'_n - \theta'_n)} \left(\frac{W_T}{W_{PL}}\right) + \frac{\alpha \, w'_n}{1 - \xi'} = 0$$

Hence

$$\frac{w'_n (1 - \xi'_n - \theta'_n)}{1 - \xi'_n} = -\frac{1}{\alpha} \frac{W_T}{W_{PL}} = \text{const.}$$

$$n = 1, 2, \ldots\ldots\ldots\ldots\ldots, N \qquad (25)$$

From (25), ξ'_1, ξ'_2, ... , ξ'_{N-} can be expressed in terms of ξ'_N

and thus

$$\left.\begin{aligned} 1-\xi'_1 &= \frac{\theta'_1 w'_1 (1-\xi'_N)}{\theta'_N w'_N + (1-\xi'_N)(w'_1 - w'_N)} \\ &\cdots\cdots\cdots \\ 1-\xi'_{(N-1)} &= \frac{\theta'_{N-1} w'_{N-1} (1-\xi'_N)}{\theta'_N w'_N + (1-\xi'_N)(w'_{N-1} - w'_N)} \end{aligned}\right\} \quad (26)$$

The above (26) equations are the optimization relationships of Subotowicz problem for N-step rocket parameters with different construction ratios $\theta'_1, \theta'_2, \ldots\ldots\ldots, \theta'_N$ and propellant exhaust velocities $w'_1, w'_2, \ldots\ldots\ldots, w'_N$.

HALL AND ZAMBELLI PROBLEM

Hall and Zambelli have analyzed the problem of optimum weight distribution for multistage rockets having different specific impulses and structural factors in each stage by a different technique. Their basic assumptions are (a) thrust direction is constant (no turning), (b) thrust is the only force acting (no drag or gravity) and (c) specific impulse is constant throughout any given stage but may vary from stage to stage. Again the notation with dashes denotes the corresponding notation of ref. (3). Evidently (14) and (16) for extremum of $\left(\frac{W_T}{W_{PL}}\right)$ can be rewritten as

$$\left.\begin{aligned} &\frac{\partial}{\partial \lambda_j}\left[\log\left(\frac{W_T}{W_{PL}}\right) + \alpha V\right] = 0 \\ \text{and} \quad &\sum_{j=1}^{N} V_j - V_c = 0 \end{aligned}\right\} \quad (27)$$

Also the following relations hold good

$$\mu_j = 1 - \beta'_j \quad (28)$$

and

$$\lambda_j = \frac{\mu'_j (1-\beta'_j)}{1-\mu'_j \beta'_j} \quad (29)$$

By assumptions of Hall & Zambelli and (28) and (29), V_j in (27) is given by

$$V_j = C'_j \log \mu'_j$$

and (27) can be written as

$$\frac{\partial}{\partial \mu'_j}\left[\log\left\{\prod_{j=1}^{N}\left(\frac{\mu'_j(1-\beta'_j)}{1-\mu'_j\beta'_j}\right)\right\} + \alpha \sum_{j=1}^{N} C'_j \log \mu'_j\right]\frac{\partial \mu'_j}{\partial \lambda_n} = 0 \quad (30)$$

and

$$\sum_{j=1}^{N} C'_j \log \mu'_j - V_c = 0 \tag{31}$$

Now $\frac{\partial \mu'_j}{\partial \lambda_n} \neq 0$ because then from (29)

$$\mu'_j = \frac{1}{\beta'_j} \quad \text{for all } j$$

and thus $\lambda_1 \rightarrow \infty$ which can be physically interpreted as a case of single stage rocket with zero payload and for this case evidently the question of optimum staging does not arise. Hence (30) gives

$$\frac{\partial}{\partial \mu'_j} \left[\log \left\{ \prod_{j=1}^{N} \left(\frac{\mu'_j \cdot (1 - \beta'_j)}{1 - \mu'_j \ \beta'_j} \right) \right\} + \alpha \sum_{j=1}^{N} C'_j \ \log \mu'_j \right] = 0 \tag{32}$$

Differentiation and simplification of (32) yields in terms of notation of ref. (3) (in which $\alpha = \lambda'$)

$$\mu'_j = (\lambda' C'_j + 1) / (\lambda' C'_j \ . \ \beta'_j); \ j = (1, 2, \ldots \ldots, N) \tag{33}$$

(31) and (33) provide a system of $(N+1)$ equations in $(N+1)$ unknowns λ' and μ'_j that provide the solution of the problem of ref. (3). Special cases of ref. (3) namely (*i*) same exhaust velocity C'_j for every stage (*ii*) same exhaust velocity and structural factor β'_j for every stage, easily follow from (31) and (33).

MALINA AND SUMMERFIELD LAW

Malina & Summerfield[4] have found out that in a dragless and field-free space optimum staging occurs for a multistage rocket with same specific impulse and structural factor for each stage when payload ratios are equal, the criterion of optimization being minimum initial rocket weight for a given velocity performance (or maximum burntout velocity for a given gross weight of rocket). This result finds easy deduction from (14) and (16) which by Malina & Summerfield assumptions transform to

$$\frac{\partial}{n} \left[\left(\frac{W_T}{W_{PL}} \right) + \alpha V \right] = 0 \tag{34}$$

and

$$\sum_{n=1}^{N} g \ I \log \frac{\lambda_n}{\lambda_n (1 - \mu) + \mu} = V_c \tag{35}$$

Equation (34) gives for λ_n and λ_{n+1}

$$\frac{1}{\lambda_n} \left(\frac{W_T}{W_{PL}} \right) + \frac{\alpha g I \mu}{[\lambda_n (1 - \mu) + \mu]} = 0 \tag{36}$$

and

$$\frac{1}{\lambda_{n+1}} \left(\frac{W_T}{W_{PL}} \right) + \frac{\alpha g I \mu}{[\lambda_{n+1} (1 - \mu) + \mu]} = 0 \tag{37}$$

(36) and (37) yield

$$\lambda_n = \lambda_{n+1}\,; \quad n = 1, 2, \ldots\ldots\ldots\ldots, (N-1) \tag{38}$$

which proves Malina & Summerfield result. It should be noted that equations (38) also hold valid for optimization of final burntout velocity when given take-off weight serves as a constraint.

APPENDIX

If ψ_1 be the thrust attitude angle, the burntout velocity V_{1b} and burntout altitude h_{1b} (measured vertically from the launch point) for a single stage rocket will be given by[8]

$$V_{1b} = V_\circ + g\, I_1 \log \frac{M_f}{M_e} - g\, t_1 \sin \psi_1 \tag{1}$$

$$h_{1b} = \left[V_\circ t_1 + g\, I_1\, t_1 \left\{ 1 - \frac{\log\left(\frac{M_f}{M_e}\right)}{\left(\frac{M_f}{M_e}\right) - 1} \right\} - \tfrac{1}{2}\, g \sin\, \psi_1\, t_1^2 \right] \sin \psi_1 \tag{2}$$

where

$$t_1 = \frac{g\, I_1\, (M_f - M_e)}{F} \tag{3}$$

and M_f is initial mass of the rocket, M_e final mass, $V_\circ$ initial velocity of the rocket and F rocket thrust. Let r_{o1} be the initial thrust to weight ratio, then

$$r_{o_1} = \frac{F}{M_f\, g}$$

Hence (1), (2) and (3) can be written as

$$V_{1b} = V_\circ + g\, I_1 \log \left[\frac{\lambda_1}{\lambda_1\,(1 - \mu_1) + \mu_1} \right] - g\, t_1 \sin \psi_1 \tag{4}$$

$$h_{1b} = V_\circ\, t_1 \sin \psi_1 + g\, I_1\, t_1 \sin \psi_1 \left[1 - \frac{\log \frac{\lambda_1}{\lambda_1(1-\mu_1)+\mu_1}}{\frac{(\lambda_1 - 1)\,\mu_1}{\lambda_1(1-\mu_1) + \mu_1}} \right]$$

$$- \frac{g \sin^2 \psi_1\, t_1^2}{2} \tag{5}$$

where

$$t_1 = \frac{I_1}{r_{o1}} \left(1 - \frac{M_e}{M_f} \right) = \frac{\mu_1(\lambda_1 - 1)\, I_1}{\lambda_1(r_{o1})} \tag{6}$$

For a two-stage rocket whose thrust attitude angle of the second stage is Ψ_2, the burntout velocity of the rocket V_{2b} will be given by the sum of the component of the burntout velocity after the first stage in the direction of Ψ_2, *i.e.* $V_{1b} \cos (\Psi_2 - \Psi_1)$ and the velocity contribution due to the second stage, *i.e.*, V_2. The burntout altitude h_{2b} will be the sum of the altitude attained due to the first stage *i.e.* h_{1b} (h_1), and that due to the second stage h_2. Hence assuming the rocket to start from rest $(V_o = 0)$

$$V_{2b} = V_{1b} \cos (\Psi_2 - \Psi_1) + V_2 = V_1 \cos \xi_1 + V_2 \tag{7}$$

where

$$V_2 = g\, I_2 \log \left[\frac{\lambda_2}{\lambda_2 (1 - \mu_2) + \mu_2} \right] - g\, t_2 \sin \Psi_2$$

$$t_2 = \frac{\mu_2 (\lambda_2 - 1)\, I_2}{\lambda_2 (r_{o2})}$$

$$h_{2b} = h_{1b} + h_2$$

where

$$h_2 = V_{1b} \cos \xi_1 \sin \Psi_2 t_2 + g\, I_2 t_2 \sin \Psi_2 \left[1 - \frac{\log \dfrac{\lambda_2}{\lambda_2 (1 - \mu_2) + \mu_2}}{\dfrac{(\lambda_2 - 1)\, \mu_2}{\lambda_2 (1 - \mu_2) + \mu_2}} \right] - \frac{g \sin^2 \Psi_2\, t_2^2}{2} \tag{8}$$

Hence from (5) and (8)

$$h_{2b} = V_1 \cos \xi_1 \sin \Psi_2 t_2 + \sum_{n=1}^{2} g\, I_n\, t_n \sin \Psi_n \left[1 - \frac{\log \dfrac{\lambda_n}{\lambda_n (1 - \mu_n) + \mu_n}}{\dfrac{(\lambda_n - 1)\, \mu_n}{\lambda_n (1 - \mu_n) + \mu_n}} \right]$$

$$- \sum_{n=1}^{2} \frac{g \sin^2 \Psi_n\, t_n^2}{2} \tag{9}$$

Similarly for a three-stage rocket

$$V_{3b} = V_{2b} \cos \xi_2 + V_3 = (V_1 \cos \xi_1 + V_2) \cos \xi_2 + V_3 = V_1 \prod_{k=1}^{2} \cos \xi_k$$

$$+ V_2 \cos \xi_2 + V_3 = \sum_{n=1}^{2} V_n \prod_{k=n}^{2} \cos \xi_k + V_3 \tag{10}$$

where V_3 and t_3 are given by similar expressions as in the case of two stage rocket

$$h_{3b} = h_{2b} + h_3$$

where

$$h_3 = V_{2b} \cos \xi_2 \sin \Psi_3 t_3 + g\, I_3 t_3 \sin \Psi_3 \left[1 - \frac{\log \dfrac{\lambda_3}{\lambda_3 (1 - \mu_3) + \mu_3}}{\dfrac{(\lambda_3 - 1)\, \mu_3}{\lambda_3 (1 - \mu_3) + \mu_3}} \right] - \frac{g \sin^2 \Psi_3\, t_3}{2} \tag{11}$$

Now by (7) the first term on the R.H.S. of (11) can be written as

$$V_{2b} \cos \xi_2 \sin \Psi_3 t_3 = \left(V_1 \prod_{k=1}^{2} \cos \xi_k + V_2 \cos \xi_2 \right) \sin \Psi_3 t_3 \tag{12}$$

Hence from (9), (11) and (12),

$$h_{3b} = \sum_{n=1}^{3} \left(V_1 \prod_{k=1}^{n-1} \cos \xi_k + V_2 \prod_{k=2}^{n-1} \cos \xi_k + V_3 \prod_{k=3}^{n-1} \cos \xi_k + \ldots\ldots + V_{n-1} \operatorname{Cos} \xi_{n-1} \right)$$

$$\sin \Psi_n t_n + \sum_{n=1}^{3} g I_n t_n \sin \Psi_n \left[1 - \frac{\log \dfrac{\lambda_n}{\lambda_n (1-\mu_n) + \mu_n}}{\dfrac{(\lambda_n - 1) \mu_n}{\lambda_n (1-\mu_n) + \mu_n}} \right] - \sum_{n=1}^{3} \frac{g \sin^2 \Psi_n t_n^2}{2} \tag{13}$$

Thus proceeding step by step the burntout velocity V and burntout altitude H of a N-stage rocket with varying thrust attitude angle from stage to stage are given by

$$V = \sum_{n=1}^{N-1} V_n \prod_{k=n}^{N-1} \cos \xi_k + V_N$$

where

$$\xi_k = \Psi_{k+1} - \Psi_k \qquad k = 1, 2, \ldots\ldots\ldots, (N-1)$$

$$V_n = g I_n \log \left[\frac{\lambda_n}{\lambda_n (1-\mu_n) + \mu_n} \right] - g t_n \sin \Psi_n$$

and

$$H = \sum_{n=1}^{N} \left(V_1 \prod_{k=1}^{n-1} \cos \xi_k + V_2 \prod_{k=2}^{n-1} \cos \xi_k + V_3 \prod_{k=3}^{n-1} \cos \xi_k + \ldots + V_{n-1} \cos \xi_{n-1} \right)$$

$$\sin \Psi_n t_n + \sum_{n=1}^{N} (g t_n I_n \sin \Psi_n \left[1 - \frac{\log \dfrac{\lambda_n}{\lambda_n (1-\mu_n) + \mu_n}}{\dfrac{(\lambda_n - 1) \mu_n}{\lambda_n (1-\mu_n) + \mu_n}} \right] - \sum_{n=1}^{N} \frac{g \sin^2 \Psi_n t_n^2}{2}$$

where

$$t_n = \frac{\mu_n (\lambda_n - 1) I_n}{(\lambda_n) r_{on}}$$

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