TORSIONAL VIBRATIONS OF FINITE ISOTROPIC COMPOSITE CIRCULAR CYLINDERS

RAM KUMAR

Defence Science Laboratory, Delhi

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The frequency equation for the torsional vibrations of composite cylinders, either concentric or bonded end to end, have been obtained. In the former case, the variation of the resonant frequency with inner cylinder thickness has been studied for the lowest symmetric and antisymmetric modes. In the latter case, the variation of the resonant frequency with change in the ratio of the lengths of the cylinders has been studied for the fundamental mode and its first harmonic in the frequency range where propagation constant for one part of cylinder is real whereas the propagation constant for the other part is imaginary. Vibration patterns for some particular cases for the latter type have been drawn.

Baltrukonis *et al*[1] discussed the axial-shear vibrations of an infinitely long composite circular cylinder with two concentric circular cylindrical layers which are perfectly bonded at their interface. The problem of composite cylinders was initiated because of its application in the design of solid propellant rockets.

In this paper is discussed the torsional vibrations of two different types of finite, isotropic, composite circular cylinders *viz.* (*i*) a hollow composite cylinder with two concentric cylindrical layers and (*ii*) a solid composite cylinder with two layers such that interface of the layers is a normal cross-section of the cylinder.

HOLLOW COMPOSITE CYLINDER

It has been assumed that the outer casing is very thin and that its stiffness is very large compared to that of the core material. This assumption is valid keeping in view our interest in the problem of solid propellant rocket. Frequency equation for this problem has been obtained by satisfying requisite boundary conditions at lateral surfaces and interface of the cylinder whereas the boundary condition at the flat ends determines the order of mode. This frequency equation has been solved for different geometries for motions which are symmetrical as well as antisymmetrical about the central plane of the cylinder and the frequency curves have been drawn for both the cases.

Statement of the problem

Consider the torsional vibrations of a finite, isotropic, composite bar (length $2c$) consisting of two concentric, thick-walled cylinders (Radii of outer casing and inner core being r_1 and r_2 respectively and interface being at the radius $r = a$) of different materials and assume that the displacements are such that

$$\left.\begin{aligned} u_r(r, \theta, z, t) &= u_z(r, \theta, z, t) = 0 \\ u_\theta &= U_\theta(r, z, t) \end{aligned}\right\} \tag{1}$$

where r, θ, z are polar cylindrical coordinates and u_r, u_θ, u_z are displacement components in these directions. The only non-zero stresses are the tangential stresses $\tau_{r\theta}$ and $\tau_{\theta z}$. In this case, the equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{\rho}{\mu}\cdot\frac{\partial^2 u_\theta}{\partial t^2} \tag{2}$$

For simple harmonic solution, we put

$$\theta = u(r) \left\{ \begin{matrix} \cos \\ \sin \end{matrix} (\gamma z) \right\} \exp (i \omega t) \tag{3}$$

where γ is the propagation constant in the axial direction and ω is the circular frequency. We shall take cos (γz) or sin (γz) according as the motion is symmetric or anti-symmetric about the central plane.

From (2) and (3) we get

$$\frac{\partial^2 u(r)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r)}{\partial r} - \frac{u(r)}{r^2} + (K^2 - \gamma^2)\, u(r) = 0 \tag{4}$$

where $K^2 = \dfrac{\omega^2 \rho}{\mu}$

The solution of (4) can be written as

$$u(r) = A\, J_1(\alpha r) + B\, Y_1(\alpha r) \tag{5}$$

where $\alpha^2 = K^2 - \gamma^2$ (6)

$$\therefore u_{\theta j} = \left[A_j\, J_1(\alpha_j r) + B_j\, Y_1(\alpha_j r) \left\{ \begin{matrix} \cos \\ \sin \end{matrix} (\gamma_j z) \right\} \right] \exp (i \omega t) \tag{7}$$

where $j = 1$ denotes quantity for outer layer and $j = 2$ denotes quantity for inner layer of the cylinder.

Boundary conditions at the lateral surfaces and frequency equation.

We now put boundary conditions such that (*a*) the inner and outer cylindrical surfaces are traction free, and (*b*) tangential stresses and displacements are continuous at the interface.

Mathematically, the above two conditions can be translated as

$$(\tau_{r\theta_1})_{r = r_1} = 0 \qquad \text{for all } z$$

$$i.e.\ A_1 J_2(\alpha_1 r_1) + B_1 Y_2(\alpha_1 r_1) = 0 \tag{8}$$

and $\gamma_1 = \gamma_2$ (9)

$$(\tau_{r\theta_2})_{r = r_2} = 0 \qquad \text{for all } z$$

$$i.e.\ A_2 J_2(\alpha_2 r_2) + B_2 Y_2(\alpha_2 r_2) = 0 \tag{10}$$

$$(\tau_{r\theta_1})_{r = a} = (\tau_{r\theta_2})_{r = a} \qquad \text{for all } z$$

$$i.e.\ \mu_1 [A_1 \alpha_1 J_2(\alpha_1 a) + B_1 \alpha_1 Y_2(\alpha_1 a)] - \mu_2 [A_2 \alpha_2 J_2(\alpha_2 a) + B_2 \alpha_2 Y_2(\alpha_2 a)] = 0 \tag{11}$$

$$(u_{\theta_1})_{r = a} = (u_{\theta_2})_{r = a} \qquad \text{for all } z$$

$$i.e.\ [A_1 J_1(\alpha_1 a) + B_1 Y_1(\alpha_1 a)] - [A_2 J_1(\alpha_2 a) + B_2 Y_1(\alpha_2 a)] = 0 \tag{12}$$

Eliminating the unknown constants A_1, B_1, A_2, B_2 from (8), (10), (11) and (12) we obtain the frequency equation as follows :

$$\begin{vmatrix} J_2(\alpha_1 r_1) & Y_2(\alpha_1 r_1) & 0 & 0 \\ 0 & 0 & J_2(\alpha_2 r_2) & Y_2(\alpha_2 r_2) \\ \mu_1 \alpha_1 (\alpha_1 a) & \mu_1 \alpha_1 Y_2(\alpha_1 a) & -\mu_2 \alpha_2 J_2(\alpha_2 a) & -\mu_2 \alpha_2 Y_2(\alpha_2 a) \\ J_1(\alpha_1 a) & Y_1(\alpha_1 a) & -J_1(\alpha_2 a) & -Y_1(\alpha_2 a) \end{vmatrix} = 0$$

i. e.

$$\begin{bmatrix} -\mu_2 \alpha_2 \{ -J_2(\alpha_1 r_1) Y_1(\alpha_1 a) + Y_2(\alpha_1 r_1) J_1(\alpha_1 a) \} \{ J_2(\alpha_2 r_2) Y_2(\alpha_2 a) \\ - Y_2(\alpha_2 r_2) J_2(\alpha_2 a) \} + \mu_1 \alpha_1 \{ J_2(\alpha_1 r_1) Y_2(\alpha_1 a) - Y_2(\alpha_1 r_1) \\ J_2(\alpha_1 a) \} \{ Y_2(\alpha_2 r_2) J_1(\alpha_2 a) - J_2(\alpha_2 r_2) Y_1(\alpha_2 a) \} \end{bmatrix} = 0 \quad (13)$$

We now assume that the outer casing of the cylinder is very thin *i.e.*, $r_1/a \to 1$ and that the stiffness of the outer casing is very large compared to that of the core material *i.e.*, $(\mu_1 / \mu_2) \longrightarrow \infty$. With these two assumptions the frequency equation (13) becomes indeterminate for $r_1 = a$ and, therefore, expanding it in Taylor's series in the neighbourhood of $(r_1/a = 1)$ and assuming that $\mu_1 = \mu_2 \Big| \left(\frac{r_1}{a} - 1 \right)$ it can be modified as follows:

$$\left\{ Y_2\left(\alpha_2 a . \frac{r_2}{a}\right) J_1(\alpha_2 a) - Y_1(\alpha_2 a) J_2\left(\alpha_2 a . \frac{r_2}{a}\right) \right\}$$

$$+ \frac{\alpha_2 a}{\gamma^2 a^2} \left\{ J_2\left(\alpha_2 a . \frac{r_2}{a}\right) Y_2(\alpha_2 a) - Y_2\left(\alpha_2 a . \frac{r_2}{a}\right) J_2(\alpha_2 a) \right\} = 0 \quad (14)$$

Boundary conditions at the flat ends of the cylinder

Let us assume that the flat ends of the cylinder are traction free surfaces *i.e.* $(\tau_{\theta z})_{z = \pm c} = 0$ for all values of r. Then $\sin(\gamma c) = 0$ (15)

if the motion is symmetric about the central plane and $\cos(\gamma c) = 0$ (16)
if the motion is antisymmetric about the central plane.

Thus the values of γc corresponding to symmetric and antisymmetric motions about the central plane are respectively given by

$$\gamma c = n\pi \quad (17)$$

where n is a positive integer
and

$$\gamma c = (2m + 1) \frac{\pi}{2} \quad (18)$$

where m is again an integer.

Numerical solutions

The frequency equation (14) is a transcendental equation involving $\alpha_2 a$, (r_2/a), γc and a/c. (17) or (18) fix the value of γc for a particular chosen value of n or m. Let us consider the symmetric mode for n=1 and anti-symmetric mode for m=0. Ratio a/c is defined by the dimensions of the cylinder. Thus for a particular cylinder and for a particular mode, (14) can be solved for r_2/a and $\alpha_2 a$. From the values of $\alpha_2 a$ and γa we can find the value of frequency parameter k_2a. Thus we can draw curves between r_2/a and k_2a. Here we have drawn curves between r_2/a and $(k_2a)/(k_2a)_0$ where $(k_2a)_0$ denotes the value of k_2a at r_2/a=0. In the present paper, have been drawn these curves for four different values of diameter-thickness ratios a/c=0·1, 0·2, 0·5 and 1·0 for symmetric as well as antisymmetric motions in Fig. 1 and Fig. 2 respectively. The values of k_2a for different values of r_2/a corresponding to symmetric and antisymmetric motions are given

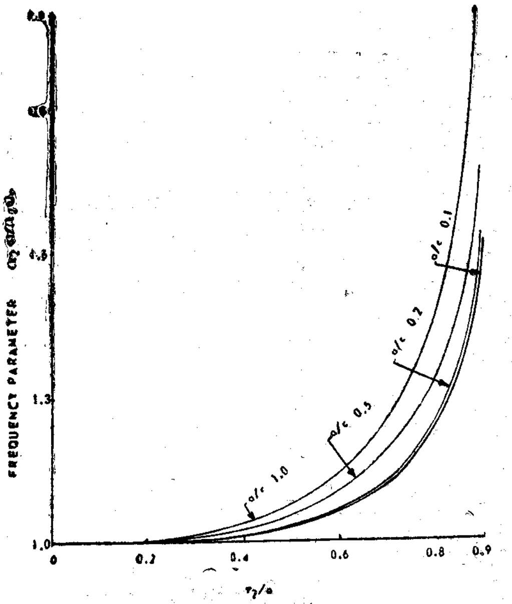


Fig. 1—Variation of frequency parameter $(k_2a)/(k_2a)_0$ with the inner radius ratio (r_2/a) for those vibrations of a cylinder which are symmetric about the central plane.

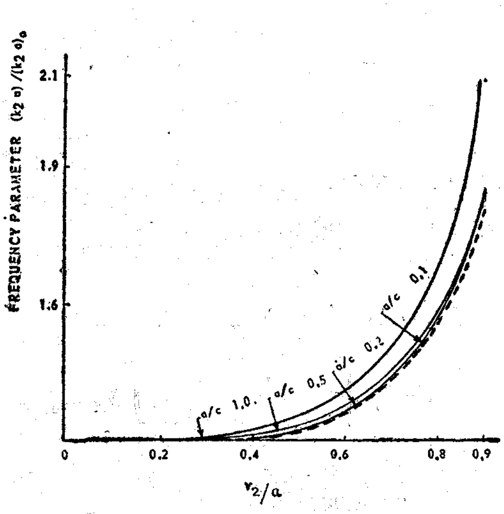


Fig. 2—Variation of frequency parameter $(k_2a)/(k_2a)_0$ with the inner radius ratio (r_2/a) for those vibrations of a cylinder which are antisymmetric about the central plane.

in Tables 1 and 2 respectively. The flatness of curves in Fig. 1 and 2 in the neighbourhood of $r_2/a = 0$ is again observed here to be similar to the flatness of curves drawn by Baltrukonis in Fig. 2 (*a*) of his paper. Further, it is observed for symmetric motions that the values of $(k_2a)/(k_2a)_0$ go on decreasing as the value of a/c decreases. In the case of antisymmetric motions the curves corresponding to $a/c = 0 \cdot 1$ and $a/c = 0 \cdot 2$ are almost coincident.

SOLID COMPOSITE CYLINDER

The torsional vibrations of a composite circular cylinder whose interface is its normal cross section, have been considered here. The frequency equation is obtained by satisfying requisite boundary conditions at the flat surfaces and the interface. Mode number is determined by boundary conditions at the lateral surface. The frequency equation has been solved for the fundamental mode and its first harmonic and the frequency spectra for these modes have been drawn graphically. Vibration patterns for particular cases have also been shown.

Let us assume that the interface of the cylinder lies at $z = 0$ and the other flat surfaces lie at $z=c$ and $z= -l$. Displacement in this case will be the same as given by (1). The solution (7) of the equation of motion will be modified as

$$u_{\theta_s} = J_1 (\alpha_s r) [A_s \cos (\gamma_s Z) + B_s \sin (\gamma_s Z)] \exp (i \omega t) \qquad (19)$$

where $s=1, 2$ stands for different materials on two sides of the interface.

We have neglected $Y_1 (\alpha_s r)$ to avoid singularities at $r = 0$.

Boundary conditions at the lateral surface

Let us assume that the lateral surface is a traction free surface. This requires that $\tau_{r\theta})_{r=a} = 0$ for all values of z.

TABLE 1

VALUES OF FREQUENCY PARAMETER (k_2a) CORRESPONDING TO DIFFERENT VALUES OF INNER RADIUS RATIO (r_2/a) FOR THAT MOTION WHICH IS SYMMETRIC ABOUT THE CENTRAL PLANE

r_2/a	Frequency parameter—k_2a			
	$a/c=0\cdot1$	$a/c=0\cdot2$	$a/c=0\cdot5$	$a/c=1\cdot0$
0·0	0·6977	1·3687	3·0188	4·6590
0·2	0·6986	1·3713	3·0196	4·6730
0·4	0·7058	1·3856	3·0883	4·8240
0·5	0·7175	1·4106	3·1739	5·0010
0·6	0·7391	1·4578	3·3227	5·3250
0·8	0·8716	1·7300	4·0979	6·9510
0·9	1·1161	2·2168	5·3803	9·8450

TABLE 2

VALUES OF FREQUENCY PARAMETER (k_2a) CORRESPONDING TO DIFFERENT VALUES OF INNER RADIUS RATIO (r_2/a) FOR THAT MOTION WHICH IS ANTISYMMETRIC ABOUT THE CENTRAL PLANE

r_2/a	Frequency parameter—k_2a			
	$a/c=0\cdot1$	$a/c=0\cdot2$	$a/c=0\cdot5$	$a/c=1\cdot0$
0·0	0·3511	0·6977	1·6861	3·0188
0·2	0·3520	0·6986	1·6957	3·0196
0·4	0·3556	0·7058	1·7091	3·0883
0·5	0·3601	0·7175	1·7412	3·1739
0·6	0·3709	0·7391	1·8031	3·3227
0·8	0·4381	0·8716	2·1524	4·0979
0·9	0·5576	1·1161	2·4976	5·3803

i.e.

$$J_o\ (\alpha_1 a)\ /\ J_1\ (\alpha_1 a) = 2\ /\ (\alpha_1\ a) \tag{20}$$

and

$$J_o\ (\alpha_2 a)\ /\ J_1\ (\alpha_2 a) = 2\ /\ (\alpha_2\ a) \tag{21}$$

Let $\alpha_1 = \alpha_2 = \alpha$ (say) be a possible solution which satisfies (20) and (21) simultaneously in which case they can be combined into a single equation

$$J_o\ (\alpha\ a)\ /\ J_1\ (\alpha\ a) = 2\ /\ (\alpha\ a) \tag{22}$$

This equation determines the value of $\alpha\, a$ for different modes of vibrations.

Boundary conditions at the flat ends and frequency equation

In this case, we again assume that flat surfaces are traction free surfaces and that surface tractions and displacement components are continuous at the interface. These boundary conditions can be written as

$$(\tau_{\theta z_1})_{z=c} = 0 \qquad \text{for all values of } r$$

i.e.

$$A_1 \sin (\gamma_1 c) - B_1 \cos (\gamma_1 c) = 0 \tag{23}$$

$$(\tau_{\theta_2 z})_{z=-l} = 0 \qquad \text{for all values of } r$$

i.e.

$$A_2 \sin (\gamma_2 l) + B_2 \cos (\gamma_2 l) = 0 \tag{24}$$

$$(u_{\theta_1})_{z=0} = (u_{\theta_2})_{z=0} \qquad \text{for all values of } r$$

i.e.

$$A_1 - A_2 = 0 \tag{25}$$

$$(\tau_{\theta z_1})_{z=0} = (\tau_{\theta z_2})_{z=0} \qquad \text{for all values of } r$$

ie.

$$B_1 \mu_1 \gamma_1 - B_2 \mu_2 \gamma_2 = 0 \tag{26}$$

Eliminating A_1, B_1, A_2, and B_2 from (23) to (26) we get the frequency equation as follows :

$$[\tan(\gamma_1 c)/\tan(\gamma_2 l)] = -(\mu_2/\mu_1)(\gamma_2/\gamma_1)$$

or

$$\frac{\tan\left\{(K_1^2-\alpha^2)^{1/2} c\right\}}{\tan\left\{(K_2^2-\alpha^2)^{1/2} l\right\}} = -\left(\frac{\mu_2}{\mu_1}\right)\frac{(K_2^2-\alpha^2)^{1/2}}{(K_1^2-\alpha^2)^{1/2}}$$

or

$$\frac{\tan\left\{\left(\omega^2 a^2\frac{\rho_1}{\mu_1}-\alpha^2 a^2\right)^{1/2}\cdot\frac{c}{a}\right\}}{\tan\left\{\left(\omega^2 a^2\frac{\rho_2}{\mu_2}-\alpha^2 a^2\right)^{1/2}\cdot\frac{l}{a}\right\}} = -\left(\frac{\mu_2}{\mu_1}\right)\frac{\left(\omega^2 a^2\frac{\rho_2}{\mu_2}-\alpha^2 a^2\right)^{\frac{1}{2}}}{\left(\omega^2 a^2\frac{\rho_1}{\mu_1}-\alpha^2 a^2\right)^{\frac{1}{2}}} \quad (27)$$

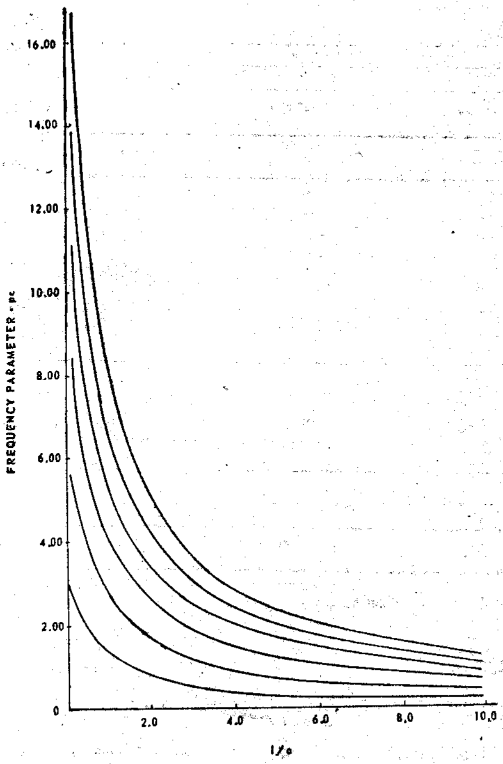


Fig. 3 —Variation of the frequency parameter pc with respect to the ratio of the lengths of copper and steel materials (l/c) for $\alpha a = 0\cdot 0$

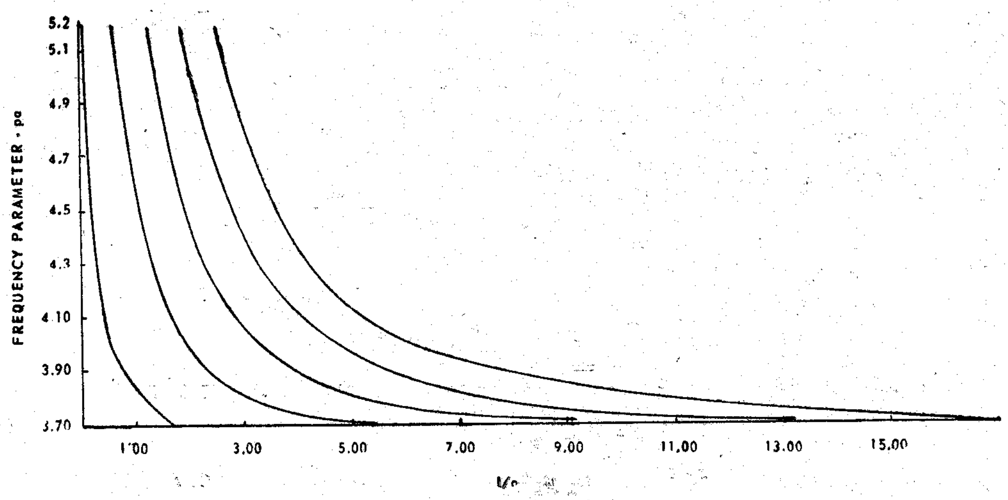


Fig. 4.—Variation of the frequency parameter pa with respect to the aspect ratio (l/a) for $\alpha a = 5\cdot 136$; $c/a = 5$

Numerical solutions

Frequency equation (27) is an equation involving ωa, αa, c/a and l/a. Corresponding values of ρ_1, ρ_2, μ_1 and μ_2 can be used according to the metals used in the composite cylinder. Let us take ρ_1, μ_1 and c for steel and ρ_2, μ_2 and l for copper. Frequency equation has been solved for these metals for the fundamental mode ($\alpha a = 0$) and its first harmonic ($\alpha a = 5\cdot136$). For the fundamental mode ($\alpha a = 0$) the above frequency equation can be written as

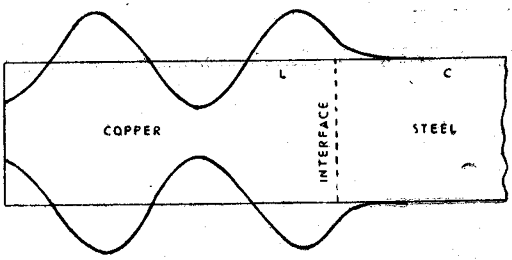
$$\frac{\tan\left\{\omega\, c\,(\rho_1/\mu_1)^{1/2}\right\}}{\tan\left\{\omega\, l\,(\rho_2/\mu_2)^{1/2}\right\}} = -\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}} \tag{28}$$

If we put $\rho_1 = 7\cdot849$ gm/cm³; $\rho_2 = 8\cdot843$ gm/cm³, $\mu_1 = 8\cdot19\times10^{11}$ dynes/cm², $\mu_2 = 4\cdot47\times10^{11}$ dynes/cm² and replace ω^2 by $p^2\times10^{11}$ then (28) can be rewritten as

$$\tan\left\{pc\,(0\cdot97897)\right\}/\tan\left\{pl\,(1\cdot40650)\right\} = -\,0\cdot78416 \tag{29}$$

After solving (29) six curves between pc and l/c have been drawn as shown in Fig. 3. These curves become asymptotic to the axis of pc as the value of pc increases. The variation of lower values of pc is quite small whereas the variation of upper values of pc is quite large.

For the first dispersive mode ($\alpha a = 5\cdot136$) we find that the propagation constant corresponding to steel becomes imaginary in the range of frequency parameter $pa = 3\cdot65157$ and $pa = 5\cdot24639$ whereas the propagation constant corresponding to copper within this range becomes real. Having substituted the values of densities, moduli of rigidity and αa, the frequency equation becomes an equation in pa, c/a and l/a. We have fixed $c/a = 5\cdot0$ and solved the resulting frequency equation for pa and l/a within the range $pa = 3\cdot70$ to $5\cdot20$. First five curves have been drawn between pa and l/a in Fig. 4. These curves are found to be asymptotic near the axis of l/a as the value of l/a increases. The variation of l/a near $pa = 3\cdot70$ is found to be more than its variation near $pa = 5\cdot20$.



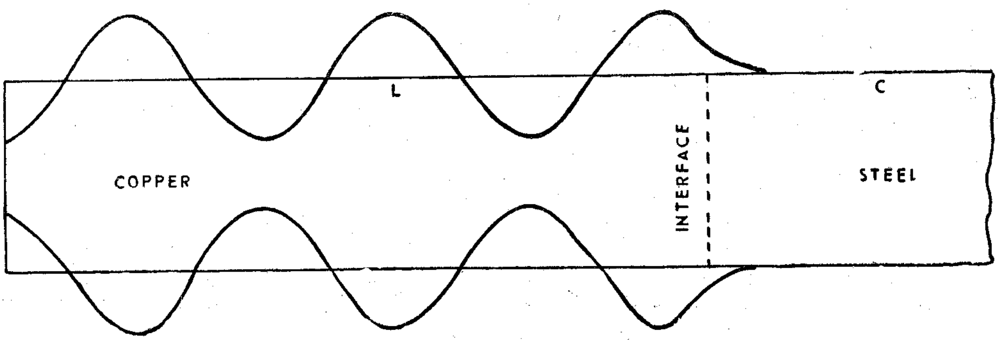


Fig. 5.—Displacements of the surfaces of the cylinders with aspect ratios $(l/a) = 4\cdot630$ and $7\cdot367$ for frequency parameter $pa = 4\cdot0$ and $c/a = 5\cdot0$

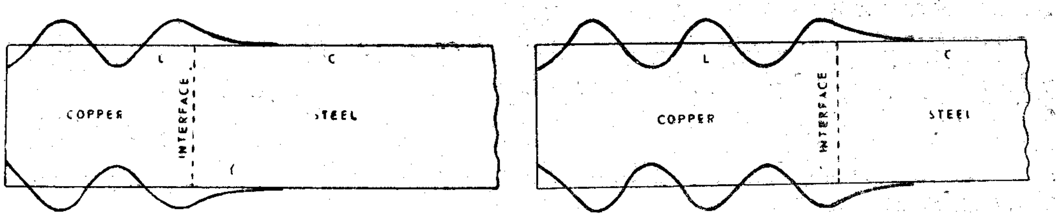


Fig. 6 —Displacements of the surfaces of the cylinders with aspect ratios $l/a = 2 \cdot 613$ and $4 \cdot 209$ for frequency parameter $pa = 4 \cdot 60$ and $c/a = 5 \cdot 00$

Corresponding to $pa = 4 \cdot 0$, $c/a = 5 \cdot 0$, vibration patterns have been drawn for $l/a = 4 \cdot 630$ and $7 \cdot 367$ in Fig. 5. Vibration patterns have also been drawn corresponding to $pa = 4 \cdot 60$, $c/a = 5 \cdot 0$ for $l/a = 2 \cdot 613$ and $4 \cdot 209$ in Fig. 6. These vibration patterns depict the attenuation of amplitudes in the metal with imaginary propagation constant as we proceed away from the interface. The amplitude ultimately vanishes at the flat end of this metal. On the other side of the interface, *i.e.* on the metal for which propagation constant is real, the amplitude varies harmonically. Amplitudes at the flat ends of this metal are more than that of the amplitude at the interface. Further, we observe that the amplitude at the flat end of this metal is inversely proportional to the frequency parameter.

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