FLOW OF POWER-LAW FLUIDS BETWEEN TWO ROTATING CO-AXIAL CONES

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(*Received* 19 *April*, 1965)

The problem of fluid flow between two coaxial cones has important applications in the theory of hydrodynamic lubrication and in the measurement of viscosity coefficients. In a recent paper Rathy[1] has discussed the flow of a Newtonian fluid between two coaxial rotating cones with a common vertex under a radial force. In the present paper an exact analytical solution of the flow behaviour of power-law fluids between two coaxial rotating cones under the action of a radial force, has been obtained.

BASIC EQUATIONS

Spherical polar coordinates are used. The common axis of cones is taken as the axis of z. The following assumptions are made:

(*i*) The radial and the transverse velocities v_r and v_θ are zero.

(*ii*) The motion is steady.

(*iii*) The motion is axially-symmetric and consequently the derivative of velocity with respect to ϕ is zero.

(*iv*) There is no transverse pressure gradient *i.e.* $\frac{\delta p}{\delta \phi} = 0$

(*v*) The external force is in the radial direction and it is of the form $\rho_0 r^l F(\theta)$

If possible let $v_\phi = r^m \sin\theta f(\theta)$ be the velocity of a fluid element at the point (r, θ, ϕ) in spherical polar coordinates and let the inner and outer cones be respectively rotating with angular velocities ω_1 and ω_2. Let α and β be the semi-vertical angles of the cones.

Since v_ϕ on the surfaces of the cones has the values $r \sin \alpha . \omega_1$ and $r \sin\beta . \omega_2$, the assumption $v_\phi = r^m \sin \theta f(\theta)$ is valid only for $m = 1$ and the only possible form of the velocity is

$$v_\phi = r \sin\theta f(\theta) \tag{1}$$

The rheological relation for power-law fluids becomes

$$\Gamma_{\theta\phi} = \mu \left| \frac{\sin\theta}{r} \frac{\delta}{\delta\theta} \left(\frac{v_\phi}{\sin\theta} \right) \right|^{n-1} \frac{\sin\theta}{r} \frac{\delta}{\delta\theta} \left(\frac{v_\phi}{\sin\theta} \right) \tag{2}$$

and other components of shear stress vanish, since $v_r = 0$ and $v_\theta = 0$

If $\rho_0 g_r$ is the external force along the radius, the equations of motion are

$$-\rho_0 \frac{v_\phi^2}{r} = -\frac{\delta p}{\delta r} + \rho_0 g_r \tag{3}$$

$$-\rho_\circ \frac{v_\phi^2}{r} \cot\theta = -\frac{1}{r} \frac{\delta p}{\delta \theta}$$

and

$$0 = -\frac{1}{r \sin\theta} \frac{\delta p}{\delta \phi} + \frac{1}{r}\left[\frac{\delta \,\Gamma_{\theta\phi}}{\delta\theta} + 2 \cot\theta \;\Gamma_{\phi\theta}\right] \tag{5}$$

Since $\frac{\delta p}{\delta \phi}$ is taken to be zero threfore, from (5) by integrating, we get

$$\Gamma_{\theta\phi} \sin^2\theta = A \tag{6}$$

which is a general result valid for all fluids under the above mentioned situations.

From (3) and (4) we see that $l = 1$, so that

$$\rho_\circ g r = \rho_\circ r F(\theta) \tag{7}$$

The flow of the fluids is now studied separately when (*i*) $\omega_2 > \omega_1$ (*ii*) $\omega_2 < \omega_1$ (*iii*) $\omega_2 = \omega_1$, and (*iv*) ω_1 and ω_2 are the angular velocities in opposite directions.

SOLUTION OF EQUATIONS

Case (*i*) $\omega_2 > \omega_1$ From (6) we see that $\Gamma_{\theta\phi}$ has the same sign throughout the annulus and then from the rheological equation

$$\Gamma_{\theta\phi} = \mu \,|\sin\theta f'(\theta)|^{n-1} \sin\theta f'(\theta) \tag{8}$$

we find that $f'(\theta)$ has the same sign as $\Gamma_{\theta\phi}$. Thus if $\omega_2 > \omega_1$, $\Gamma_{\theta\phi}$ and $f'(\theta)$ must be positive and (8) has the form,

$$\Gamma_{\theta\phi} = \mu[\sin\theta f'(\theta)]^n \tag{9}$$

and $f(\theta)$, the angular velocity of the fluid element, will steadily increase from ω_1 to ω_2 as θ increases from α to β.

Thus from (9), on putting the values of $\Gamma_{\theta\varphi}$ from (6), we get

$$f'(\theta) = \left(\frac{A}{\mu}\right)^{\frac{1}{n}} \operatorname{cosec}^{\frac{2}{n}+1} \theta$$

with the boundary conditions

$$f(\theta) = \omega_1 \qquad \text{when } \theta = \alpha$$

and

$$f(\theta) = \omega_2 \qquad \text{when } \theta = \beta$$

On integration we get

$$\frac{f(\theta) - \omega_1}{\omega_2 - \omega_1} = \int_\alpha^\theta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \Big/ \int_\alpha^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \tag{10}$$

and from (9) .

$$\overline{\lceil \theta \phi} = \mu \operatorname{cosec}^2 \theta \, (\omega_2 - \omega_1)^n \Big/ \left[\int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \right]^n \tag{11}$$

Case (*ii*) $w_2 < w_1$. The sign of $f'(\theta)$ in this case is negative and, therefore, the rheological relation $\overline{\lceil \theta \phi} = -\mu(-\sin\theta f')^n$ holds good and on being combined with (6), gives

$$\frac{f(\theta) - \omega_1}{\omega_1 - \omega_2} = -\int_a^\theta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \Big/ \int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \tag{12}$$

which shows that the velocity in this case steadily decreases from ω_1 to ω_2 and

$$\overline{\lceil \theta\phi} = -\mu \, (\omega_1 - \omega_2)^n \operatorname{cosec}^2 \theta \Big/ \left[\int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \right]^n \tag{13}$$

Case (*iii*) $\omega_1 = \omega_2$. We see from (10) or (12) that

$$f(\theta) = \omega_1 \tag{14}$$

and

$$\overline{\lceil \theta \phi} = 0 \tag{15}$$

which shows that the whole apparatus, together with the fluid under this case, rotates about the axis of the cones as a solidified substance and no couple is needed to sustain the rotation of the cones.

Case (iv) ω_1 is opposite to the positive direction of ϕ. By putting ω_1 negative in (10) we get

$$\frac{f(\theta) + \omega_1}{\omega_2 + \omega_1} = \int_a^\theta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \Big/ \int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \tag{16}$$

Similarly from (11) we get

$$\overline{\lceil \theta \phi} = \mu \, (\omega_2 + \omega_1)^n, \operatorname{cosec}^2 \theta \Big/ \left[\int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \right]^n \tag{17}$$

Since $f(\theta)$ in this case is negative for $\theta = a$ and positive for $\theta = \beta$ and since $f(\theta)$ is a continuous function, therefore $f(\theta) = 0$ for some value of θ which can be evaluated from

$$\omega_1/(\omega_1 + \omega_2) = \int_a^\theta \operatorname{cosec}^{-\frac{2}{n}+1} \theta \, d\theta \Big/ \int_a^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \tag{18}$$

Evaluation of $\rho_o \; g \, r$, p and the couple needed to keep the cones in steady rotation

From (3) and (4) by eliminating p we get

$$\frac{\delta}{\delta\theta}\left[\rho_o \, gr + \rho_o \, r \sin^2 \theta \, f^2(\theta)\right] = \rho_o \frac{\delta}{\delta r}\left[r^2 \sin \theta \cos \theta \, f^2(\theta)\right]$$

where $\rho_o\, gr$, as proved above, is $\rho_o\, r\, F(\theta)$ and $f(\theta)$ is known from (10), (12), (14) and (16)

$$\therefore \quad F(\theta) + \sin^2\theta f^2(\theta) = B + 2\int \sin\theta\cos\theta f^2(\theta)\, d\theta$$

$$i.e.\ F(\theta) = B - 2\int \sin^2\theta f(\theta) f'(\theta)\, d\theta$$

which gives $F(\theta)$, *i.e.* the external force $\rho_o\, rF(\theta)$ along the radius, for $f(\theta)$ is known already.

Since B is an arbitrary constant so let it be zero for its any other value will simply affect the values of p.

$$\therefore \quad F(\theta) = -2\int Sin^2\theta f(\theta) f'(\theta)\, d\theta \tag{19}$$

Since $f'(\theta)$ is positive when $\omega_2 > \omega_1$, the radial force should be applied towards the vertex but when $\omega_2 < \omega_1$ then $f'(\theta)$, because of being negative, will mean the radial force in the direction away from the vertex.

Again from (3) and (4) on integrating we get

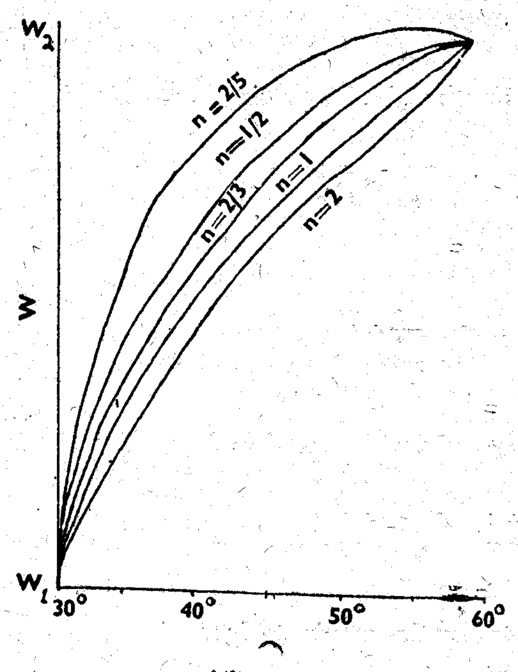

FIG. 1—Flow curve $\frac{f(\theta) - \omega_1}{\omega_2 - \omega_1}$ (when $\omega_1 < \omega_2$ and semi-vertical angles of bounding cones are 30° and 60°).

$$p = \rho_o \frac{r^2}{2}\left[F(\theta) + \sin^2\theta f^2(\theta)\right] + p_o \tag{20}$$

Lastly the couple on the outer cone

$$G_2 = \int_0^L (2\pi\, r\, Sin\,\beta\, dr\ \overline{|\theta\,\phi})\, r\sin\beta$$

$$= \frac{2\pi L^3}{3}\mu(\omega_2 - \omega_1)^n.\ 1 \Big/ \left[\int_\alpha^\beta \operatorname{cosec}^{\frac{2}{n}+1}\theta\, d\theta\right]^n \tag{21}$$

Similarly we can show that the couple G_1 on the inner cone has also the same magnitude if the slant height L of the cone is kept unchanged *i.e.*

$$G_1 = G_2$$

By taking $(\frac{2}{n}+1)$ equal to a positive integer we can evaluate all the integrals occurring in equations from (10) to (21) and so the values of ω, $\overline{|\theta\,\phi}$, $\rho_o\, gr$, p and G can be known, and by computing their values thus obtained for a set of values of n we can foresee the flow behaviour of different flows.

NUMERICAL ILLUSTRATION

Let us take the cones of semi-vertical angles 30° and 60° and from the result (10) evaluate the values of $\frac{f(\theta)-\omega_1}{\omega_2-\omega_1}$ and 55° for a set of values of n. when $\theta = 35°, 45°$ The values thus obtained are shown in Table 1. Their graphical representation is given in Fig 1.

TABLE 1

Values of $\frac{f(\theta)-\omega_1}{\omega_2-\omega_1}$ for a set of values of θ and n

θ	n				
	2	1	2/3	1/2	2/5
35°	·262	·32	·38	·438	·48
45°	·634	·69	·7548	·8	·954
55°	·85	·926	·942	·95	·97

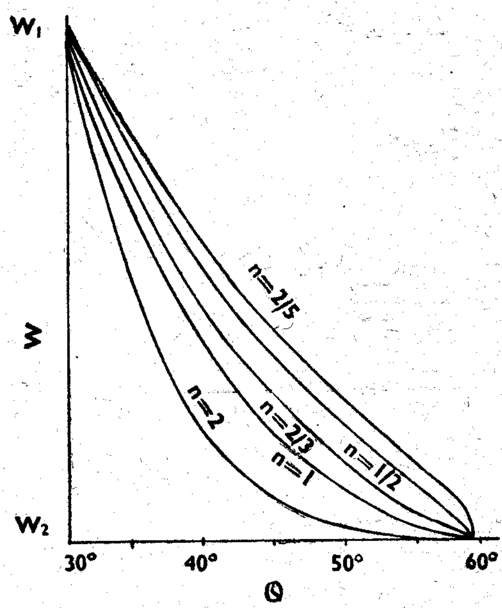

Fig 2.—Flow curve $\frac{f(\theta)-\omega_1}{\omega_2-\omega_1}$ (when $\omega_2 < \omega_1$ and semi-vertical angles of bounding cones are 30° and 60°).

Again by taking the sign of the values given in Table I as negative, we get various values for the equation (12), *i.e.* for the case when $\omega_2 < w_1$ and so the velocity graph in this case is as given in Fig 2.

From Fig 2. we see that as n decreases the angular velocity near the inner cone rises or falls more steeply than near the outer cone $\left|\frac{d\omega}{d\theta}\right|$, *i.e.* $\left|f'(\theta)\right|$ is very much larger near the inner cone than near the outer cone and accordingly the shear stress per unit area near the inner surface is much larger near the inner cone than near the outer cone.

As in the case of angular velocity the various values of the couple G, defined by equation (21), are given in Table 2.

TABLE 2

VALUES OF G FOR A SET OF VALUES OF n

$n \rightarrow$	2	1	2/3	1/2	2/5
$\frac{G/2\pi L^3\mu}{3(\omega_2-\omega_1)^n}$	·75	·28	·24	·238	·233

FLOW BETWEEN TWO ROTATING CYLINDERS

The flow is deduced by putting $\sin\theta = R/r$, $\sin\alpha = R_1/r$ and $\sin\beta = R_2/r$ in the equations (10)—(21) and then by obtaining the limit as $r\rightarrow\infty$. We can study the motion of the power law fluid between two rotating coaxial cylinders of radii R_1 and R_2 when they rotate with angular velocities ω_1 and ω_2 respectively.

Putting these values in (10) we get

$$\frac{f(\theta)-\omega_1}{\omega_2-\omega_1} = \int_{R_1}^{R} (R/r)^{-\left(\frac{2}{n}+1\right)} \frac{dR}{r\sqrt{1-\frac{R^2}{r^2}}} \Bigg/ \int_{R_1}^{R_2} \left(\frac{R}{r}\right)^{-\left(\frac{2}{n}+1\right)} \frac{dR}{r\sqrt{1-\frac{R^2}{r^2}}}$$

$$= \int_{R_1}^{R} R^{-\left(\frac{2}{n}+1\right)} \left(1+\frac{1}{2}\,\frac{R^2}{r^2}+\ldots\right) dR \Bigg/ \int_{R_1}^{R_2} (R)^{-\left(\frac{2}{n}+1\right)} \left(1+\frac{1}{2}\,\frac{R^2}{r^2}+\ldots\right) dR$$

$$= \int_{R_1}^{R} R^{-\left(\frac{2}{n}+1\right)} dR \Bigg/ \int_{R_1}^{R_2} R^{-\left(\frac{2}{n}+1\right)} dR \qquad (22)$$

when $r \rightarrow \infty$

$$\text{Therefore} \quad \frac{f(\theta)-\omega_1}{\omega_2-\omega_1} = \frac{R_2^{\,2/n}}{R^{2/n}} \cdot \frac{R^{2/n}-R_1^{\,2/n}}{R_2^{\,2/n}-R_1^{\,2/n}}$$

If we put $R_1 = R_2 K$ and $R = R_2\rho$, then $\frac{f(\theta)-\omega_1}{\omega_2-\omega_1} = \frac{1}{\rho^{2/n}} \cdot \frac{\rho^{2/n}-K^{2/n}}{1-K^{2/n}}$ (23)

which gives the angular velocity between $\rho = K$ to $\rho = 1$

Similarly from (11), we have

$$\Gamma_{\theta\phi} = \frac{\mu\,(R/r)^{-2}(\omega_2-\omega_1)^n}{\left[\int_{R_1}^{R_2} (R/r)^{-(2/n+1)} \frac{dR}{r\sqrt{1-R^2/r^2}}\right]^n}$$

$$= \mu\,(\omega_2-\omega_1)^n \left(\frac{2}{n}\right)^n \frac{1}{R^2} \cdot \frac{R_1^{\,2} R_2^{\,2}}{[R_2^{\,2/n}-R_1^{\,2/n}]^n}$$

$$= \left(\frac{2}{n}\right)^n . \mu (\omega_2 - \omega_1)^n . \frac{1}{\rho^2} . \frac{K^2}{[1-K^{2/n}]^n} \tag{24}$$

which shows that the shear stress at any point varies inversely as the square of its distance from the axis of the cylinders.

Lastly, from the equation (21) by integrating its integrand from r to $r+h$, under the above conditions, we can obtain the magnitude of the couple needed to sustain the steady angular velocity w_2 of the outer cylinder and ω_1 of the inner cylinder.

Thus

$$G_2 = \frac{2\pi\mu}{3} . (\omega_2 - \omega_1)^n \left[(r+h)^3 - r^3 \right] . 1 \Big/ \left[\int_\alpha^\beta \operatorname{cosec}^{\frac{2}{n}+1} \theta \, d\theta \right]^n$$

$$= \frac{2\pi\mu}{3} . (\omega_2 - \omega_1)^n \left[(r+h)^3 - r^3 \right] . 1 \Big/ \left[\int_{R_1}^{R_2} \left(\frac{R}{r}\right)^{-\left(\frac{2}{n}+1\right)} \frac{dR}{r\sqrt{1-\frac{R^2}{r^2}}} \right]^n$$

$$= \frac{2\pi\mu}{3} (\omega_2 - \omega_1)^n 3h \left(\frac{2}{n}\right)^n \frac{R_1^2 R_2^2}{(R_2^{2/n} - R_1^{2/n})^n}$$

$$= \left(\frac{2}{n}\right)^n 2\pi\mu (\omega_2 - \omega_1)^n h . K^2/[1 - K^{2/n}]^n \tag{25}$$

It can be seen that the magnitude of the couple on the inner cylinder has also the same value.

Again from (19) we get

$$F(\theta) = -\frac{2}{r^2} \int_{R_1}^{R_2} \left[R^2 \left\{ \omega_1 + \frac{\omega_2 - \omega_1}{\int_{R_1}^{R_2} R^{-(2/n+1)} dR} \int_{R_1}^{R} R^{-(2/n+1)} dR \right\} \right.$$

$$\times \frac{(\omega_2 - \omega_1) R^{-(2/n+1)}}{\int_{R_1}^{R_2} R^{-(2/n+1)} dR}$$

$$= 0 \tag{26}$$

in the limit as $r \to \infty$.

Similarly from the equation (20) by putting the values of $F(\theta)$, $f(\theta)$ and $\sin\theta = R/r$ and taking the limit as $r \to \infty$ we get

$$p = \frac{\rho_0}{2} r^2 . \frac{R^2}{r^2} \left\{ \omega_1 + \frac{\omega_2 - \omega_1}{\rho^{2/n}} . \frac{\rho^{2/n} - K^{2/n}}{1 - K^{2/n}} \right\}^2 + p_0$$

$$= p_0 + \frac{\rho_0 R^2}{2} \left[\omega_1 + \frac{\omega_2 - \omega_1}{\rho^{2/n}} . \frac{\rho^{2/n} - K^{2/n}}{1 - K^{2/n}} \right]^2 \tag{27}$$

Similarly the flow for a few other non-Newtonian fluids can also be derived.

ACKNOWLEDGEMENTS

I am extremely grateful to Prof. J. N. Kapur, Head of Mathematics Department, I.I.T., Kanpur for his inspiring guidance in the preparation of this paper.

REFERENCE

1. RATHY, R. K., ZAMM, XLIII, 283, (1963)