RECENT DEVELOPMENTS IN ROCKET FLIGHT OPTIMIZATION PROBLEMS

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(*Received* 3 *May* 1966)

This communication reviews the recent work done in the field of rocket flight and design optimization problems indicating the various mathematical techniques in common use for solving such types of problems. The present trend of research and the newer techniques being developed have also been pointed out.

Problems in rocket flight can be broadly classified into two categories: (*i*) cases where the rocket characteristics are supposed to be known and its mission or performance is to be studied (*ii*) cases where the mission or performance requirements are known in advance and the problem is to design a rocket to satisfy them. Both these categories lead to the problems of optimization which may be designated as "Optimal design" and "Optimal control" problems. The main problem is to find out the trajectory that a rocket should follow and the way it should be steered so that the value of the quantities called the "payoff" or the "performance index" may be maximum or minimum. These quantities may be 'range', 'altitude', 'payload', 'time of flight', 'cost', 'placing of rocket vehicle of a maximum weight into a given orbit' etc. In some problems the quantity to be extremised depends directly on the values of variables and may be treated by employing the theory of maxima and minima. There are also situations where the quantity to be extremised depends upon the history of one or more functions called the functionals. Such problems are more generally treated by the discipline of the calculus of variations. Though some newly developed methods *e.g.* 'dynamic programming', 'theory of linear integrals by the Green's theorem', 'gradient and steepest ascent' have been successfully employed but the method of variational calculus is most commonly used. Most of these analytical methods give only the necessary conditions for optimality and many of the solutions existing in literature satisfy only the necessary conditions. Work for obtaining sufficient conditions has been taken up recently and efforts are afoot for obtaining existence theorems for optimality.

OPTIMAL DESIGN PROBLEMS

In the optimal designing problems of rocket vehicles the factors which can be controlled are: mass distributions; shape and other design parameters such as area of nozzle; the length of the motor etc. In multiple stage rocket problems, the oldest one in this category, the method of staging was suggested to overcome the difficulty of attaining high velocities with a single stage rocket. The distribution of the mass at different stages was so arranged that a given payload achieved a maximum all-burnt velocity. By using the ordinary maxima and minima, this problem was first solved by Malina & Summerfield[1] under the restricted conditions: (*a*) the flight path is vertical and in vacuum (*b*) acceleration due to gravity is neglected and (*c*) specific impulses and structural factors are constant and same for every stage. It was shown that for optimum solution the mass ratios of each step are equal to each other. Vertregt[2,3] extended the solution to the case where the stages have equal specific impulses only. Goldsmith[4] further extended the results to the case of two-stage rockets having different specific impulses and structural factors. Another analysis was presented by Weisbord[5]. Coleman[6] further showed that a better optimization analysis would be by including a scaling law for structural factor which may account for its variation with step size. The case in which the structural mass of a step is related linearly to stage propellant mass was treated by Chase[7]. This relation was determined by actual preliminary design studies considering a given engine combination and a series of propellant loadings.

At present the emphasis on vehicle sizing has shifted from seeking minimum initial gross weight to minimum initial overall system cost. Uptil now it was thought that a minimum ratio of initial gross weight to payload weight means a minimum cost vehicle which, however, is not true. The problem of minimising the cost was solved by Goldbaum & White[8]. Other solutions[9,10] also exist. Formulating another solution to the cost optimization problem, Arthur Mager[11] has recently proved that for reasonably similar stages the cost optimization is not drastically different from weight optimization. However, when one of the stages is recoverable and thus has much lower effective cost than its neighbours, the cost optimization design is radically different from the weight optimisation design.

When the trajectory is no longer rectilinear and aerodynamic forces are included, the problem leaves the realm of ordinary calculus and enters into the field of calculus of variations. In this connection mention may be made of the work done by Cavoti[12,13] Mason *et al* [14] and Kosmodemianskii[15]. Liang-Tsen Fan *et al*[16] applied the discrete maximum principle to obtain general solutions to some optimization problems connected with multistage vehicle and Lobowe[17] used the method of dynamic programming in another solution of the problem. Recently Alford & Lear[18] used Denbow transformation to remove the physical discontinuities implicit in the multistage ballistic missile system and then applied the generalised Newton-Raphson method for the purpose of optimization. A computer programme has been given to optimise a two stage vehicle in drag environment using the two dimensional equations of motion. The angle of attack subject to an inequality constraint is used as a control variable.

Another class of problem treated by Goldsmith[19] was to find out the optimum area ratio (exit area/throat area) of a rocket nozzle which will give the maximum velocity increment of a stage. Optimum area ratio is important because with its increasing value the specific impulse increases leading to improved performance but at the same time resulting in increased weight of nozzle thereby retarding the performance. Similarly another simple problem of calculating the payload mass which will give maximum kinetic energy for a rocket of fixed structural and propellant weights was handled by Cole and Marrese[20]. Vandenkerckhove[21] solved a problem of different nature described thus: to design a solid propellant rocket when the variation of thrust with time is specified and the choice of motor dimensions, chamber pressure and propellant nature is left, within limits, with the designer; it is required to determine which chamber dimensions and pressure will optimise the design of solid propellant motor which must deliver specified thrust during the burning time. For simplicity the discussion was confined to the case of neutral and cylindrical side burning grains.

OPTIMAL CONTROL PROBLEMS

In dealing the optimal control problems the factors which can be controlled, are: (*i*) thrust magnitude and/or its direction (*ii*) the aerodynamic forces (the angle of attack). One of the earliest problems in this class, proposed and solved by Goddard[22], was to find out the optimum velocity variation throughout the powered flight of a vertically ascending rocket (sounding rocket) in atmosphere which will enable a given payload to attain a specified height with a minimum load of fuel. The optimum velocity variation was obtained by controlling the thrust magnitude in an appropriate manner throughout the powered flight. Later some more scientists particularly Tsein and Evans[23], Leitmann[24] and Miele and Cavoti[25] worked on this problem by making use of variational technique. Leitmann[26] showed that if no restriction is placed on the magnitude of thrust then for optimality, there should be an initial impulsive motion followed by a coastal flight. If, on the other hand, thrust is required to be within prescribed limits then Miele[25] proved that for a finite duration the initial flight must take place with maximum available thrust. Halkin[27] developed a numerical scheme based on the method of convex ascent for the solution of the sounding rocket problem. Garfinkel[28] obtained sufficiency conditions for the optimal solution while Ewing & Haseltine[29] proved the existence theorem.

For the case of horizontal rectilinear flights, a typical problem is that of achieving maximum range with a prescribed fuel consumption when the end speeds and the angle of attack are so adjusted that the aerodynamic lift is balanced by the instantaneous weight of the rocket. Hibbs[30] and Cicala & Miele[31] used the variational technique for the solution of this problem while Miele[32] gave an alternative approach by making use of Green's theorem. Miele & Cavoti[33] extended the solution of finding the optimum thrust programming to the case of inclined rectilinear paths.

In the case of two dimensional trajectory, the problem of achieving a maximum range was treated by a number of workers under different conditions. As the range during powered flight is very much less than that during the coastal flight and since most of the coasting takes place in the upper atmosphere the problem can be simplified by maximizing the coasting range over the flat earth with the initial velocity equal to the velocity at the cut-off[34]. Trenkle[35,36] used the powered flight parameters in a semi-empirical way to solve the maximum range problem. In case the powered flight is directly included, another type of problem can be that of finding the optimum direction of thrust of given magnitude so that the resulting range is maximum. Lawden[37] treated this problem for the case of constant thrust acceleration with zero initial velocity and flight in vacuum. Fried & Richardson[38] gave an alternative approach to the problem which was extended by Leitmann[39] for the case of flight in atmosphere over a spherical non-rotating earth. Later Leitmann[40] gave another derivation of the results of Fried & Richardson and showed how the necessary conditions for a local maximum are met. The main conclusion derived by all these workers was that for achieving maximum range with prescribed thrust the direction should remain constant throughout the powered flight. Fried[41] extended the results by relaxing the condition of constant acceleration due to gravity by retaining first order terms in the Taylor's series expansion. In his solution Lawden[42] accounted for the earth's rotation and sphericity and showed that for long range missions considerable increase in range is obtained by firing in the direction of earth's rotation. Lawden[43] further examined this problem for flight in atmosphere over a flat earth by taking aerodynamic forces smaller than the vehicle's weight and supposing the thrust to act along the longitudinal axis of the rocket. It was concluded that one may expect to offset the detrimental effect of aerodynamic drag by properly programming the angle of attack *i.e.* by employing the compensating effect of lift. For flight in vacuum between prescribed positions and velocities in a general gravitational field, Lawden[44,45] showed that the requirements on fuel were minimum if the trajectory was described by portions having zero thrust and impulsive thrust. For a given number of impulses, Lawden derived a criterion for (*i*) selection of internal points along the flight paths where impulsive thrusts are to be applied, (*ii*) magnitude and direction of these impulses.

Pioneering work was done by Breakwell, Fried, Leitmann, Miele, Newton etc., who tried to give the general properties of optimum trajectories considering the rocket as a variable point mass. Leitmann[46] solved the general problem of extremising a "payoff" when the flight took place with "bounded mass flow rate" in a constant gravitational field as well as in a field where the potential is a quadratic function of position. It was proved that for extremum, the flight should take place either at maximum or zero flow rates and at the most three such regimes can arise. The criterion for the appropriate mass flow was derived (the case where there is no restriction on the mass flow rate was treated by Fried[47]. For flight in constant gravitational field it was proved that the optimum mass flow consists of at the most three portions flown either at zero or maximum rate. Miele[48] extended the problem to the case where flight takes place in atmosphere and demonstrated that mass flow consists of phases which can change discontinuously among zero, variable and maximum flow rates but did not establish the criterion for the selection of the proper flow regimes. This aspect of the problem was treated by Breakwell[49]. Knowing the mass flow rate, it is comparatively easy to establish the optimum programming of thrust direction or angle attack. Newton[50] found out such a programme with the help of variational technique. In general, for quasi-steady flight, Breakwell[49] showed that optimum thrust direction is given by $\epsilon = \tan^{-1} \dfrac{\partial D}{\partial L}$ while Miele[48] proved that as far as thrust modulus is concerned the extremal arc is discontinous and consists of subarcs of three types : (*a*) null thrust arc $T = 0$ (*b*) maximum thrust arc $T = T_{max}$ and (*c*) continuously varying thrust arc. The way in which these different subarcs are combined depends upon the nature of the "payoff" to be extremised and the boundary conditions of the problem. As pointed out earlier the same conclusions hold for non-steady flight as for quasi-steady flight. Thus, in particular, in the simple problem of minimisation of propellant consumption in the case of a vertical ascending rocket for given end values of the velocity and altitude, the flight time being free, the programming of the thrust must be divided into three phases[23,51] —coasting, maximum thrust and variable thrust. The way, in which these regions are combined, depends upon the boundary conditions of the problem *e.g.* if both initial and final velocities are zero as in ascending rocket, the initial flight is with maximum thrust, the intermediate flight with variable thrust and final flight with zero thrust. Again, in the case of two dimensional flight in vacuum with no lift, Lawden[52] proved that the optimum thrust direction with respect to horizon is a bilinear function of time. Regarding the thrust modulus, the extremal arc consists of only two kinds, coasting subarcs and maximum thrust subarcs [53,55]. As pointed out earlier Leitmann[46] actually proved that the extremal path is composed of no more than three subarcs. Again the combination of these subarcs into a single extremal arc depends upon the nature of the function to be extremised. Thus in the particular problem say of maximisation of all-burnt velocity; knowing the propellant mass, initial velocity zero, given final altitude the inclination of the final velocity is zero and range is free, Fried[56] concluded that the trajectory is composed of maximum thrust subarcs followed by a coasting subarc. Along the maximum thrust subarc the thrust direction is a linear function of time. Similarly for the case when the range is to be maximum for a given mass of propellant, initial velocity zero, initial and final altitudes equal, the velocity modulus, the path inclination at the final point and the time assumed free, Fried & Richardson[38] and Newton[50] proved that the extremal arc consists of two parts, the initial one to be flown with maximum thrust and the final with zero thrust. In the first subarc the thrust is inclined at a constant angle with respect to the horizon and is perpendicular to the velocity at the final point. In the case of vertical flight for given propellant mass and end velocities, the increase in altitude being free, it can be shown that the flight time is independent of the mode of propellant consumption. Also Miele[57] showed that in case of non-steady flight over a spherical earth the general conclusions regarding optimum thrust direction and optimum thrust magnitude are identical to those for the flat earth case.

In another class of problems it is required to find out the thrust magnitude programming and its direction in order to attain a maximum horizontal injection speed at a given height for known values of mass-ratio (initial mass at burnout), the exhaust and initial velocity. In case when there is no limitations on the thrust magnitudes Okhotimskii & Eneev[53] proved that (*i*) the optimum trajectory is composed of at the most three portions, either impulsive or null thrusts and (*ii*) if initial velocity is zero and the total time to injection is free, the optimum trajectory consists of only two portions, *i.e.*, an initial impulse followed by coasting flight to the injection height. Similar results were obtained by Leitmann[46] for the case of bounded thrust, the only change is a flight with maximum thrust replacing the impulsive flight. Okhotismskii & Eneev[53] had also shown that if the acceleration due to thrust is a prescribed function of time when flight takes place in vacuum and constant gravitational field, the optimum thrust direction must be a linear function of time for the maximum injection speed at a prescribed height.

ENERGY SEPARATE OR POWER LIMITED ROCKETS

It is well known that in a chemically powered rocket there are limitations to the available thrust (*i.e.* mass flow rate), while for the energy separate rocket where the fuel is not used as a propellant but only to produce thrust, the limitation is on available power and possibly on available energy. Therefore power-limited rockets are capable of producing only low thrusts[58] and are used primarily for flight outside the atmosphere. The thrust magnitude can be varied by varying both exhaust speed and mass flow rate. But if power is held constant, mass flow rate and exhaust speed cannot be varied independently. In such cases, the components of thrust acceleration may be employed as control variables Langmuir[59] showed that if exhaust speed is constant throughout the flight in a straight line then there is an optimum value for it depending upon the specified power, the velocity to be gained and time available for the acceleration. Leitmann[60] showed that the operation at maximum propulsive power is optimal for all "performance index" or "payoff".

One of the simplest problems in energy limited rockets is to find out how, for given initial and final masses and velocities, the exhaust speed should vary in a free space flight so that the energy expenditure is minimum. Ulam[61] demonstrated that the optimum exhaust speed should vary as $(c - c_o) = (v - v_o)$ where $c_o = (v_b - v_o) \Big/ \left(\frac{m_o}{m_b} - 1\right)$

Since power limited rockets are incapable of producing high thrust, transfer in space means long flight time and therefore it is of interest to achieve minimum transfer time rather than minimum energy expenditure. Preston—Thomas[62,63] established that for rectilinear transfer of a rocket in field space between positions of rest, when the flight takes place at constant acceleration and coasting, the optimum programme consists of three operations: (*i*) an initial flight at constant acceleration (*ii*) another flight at zero acceleration (coasting) and (*iii*) finally a flight at constant deceleration. Leitmann[64,65] further showed that there can still be another saving of transfer time by 4% if (*i*) the restriction on the flight at constant acceleration is removed and replaced by an acceleration depending linearly on time and (*ii*) the transfer is executed by using the maximum available power. Leitmann[65] also pointed out the disadvantages of the optimum acceleration programme as compared to the constant acceleration programme.

Since $P(t) < P$ (available), where $P(t) = -\frac{1}{2}\dot{m}c^2$ where c is exhaust speed and $\dot{m}$ is the mass flow rate, it can be shown that if $P(t)$ is as large as possible, *i.e.* equivalent to available power, the payload mass may be maximised by minimising[66] the integral $\int_0^t a^2 dt$ subject to initial and final conditions imposed on the trajectory, 'a' being the acceleration due to thrust. The problem of determining an optimum thrust acceleration programme in order to find a maximum payload was treated by Irwing & Blum[66]. Saltzer & Fetheroff[67] also solved this problem by using Hadamard's method of gradients. Recently Mehta & Rao[68] treated the problem in a most general form and proved that in the simple case where gravity variation and aerodynamic forces are ignored, a constant thrust acceleration programme achieves the mission of maximising the payload. In the presence of aerodynamic drag of the form kv^2 they solved the equations of optimum trajectory by employing perturbation technique.

In the class of power-limited rocket problems connected with escape trajectories, transfer between neighbouring orbits or planetary orbits and corrections of orbit elements were considered by (*i*) Eblbaum[69] for the case of quasi-steady flight (*ii*) Newton[70] for the case of constant exhaust speed (*iii*) Preston-Thomas[71] for the case when the thrust acceleration is constant; (*iv*) Lindorfer & Moyer[72] by using gradient theory, (*v*) Saltzer & Fetheroff[62] by applying Rayleigh-Ritz method and (*vi*) Leitmann[73] by making use of the invariance arguments.

CONCLUSIONS

The above are some of the many aspects of the optimization problems in rocket ballistics and as remarked earlier there has been constant work going on especially in using different theoretical and numerical techniques for solving such types of problems. Dreyfus[74] has shown that many of the known results in the literature can easily be obtained by the method of dynamic programming techniques. Similarly the application of gradient theory and steepest ascent method for solving optimal programming problems has been demonstrated by Kelley[75] and Bryson[76,77] respectively. McGill & Kenneth[78] have shown how the technique of Newton-Raphson iterative process can find application in such problems while the adjoint method has been applied by Jurovies & Mc. Intire[79]. Even statistical methods have found application in solving problems of optimization as demonstrated by Breakwell.[80] Theories are currently being developed that determine the optimal decision in the neighbourhood of the optimal trajectory. These involve second order analysis including, in the classical case, the second variation [81,82].

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