A NOTE ON A NORMAL DISTRIBUTION WITH VARYING STANDARD DEVIATION

Mahabaleshwara Holla and Samir Kumar Bhattacharya

Defence Science Laboratory, Delhi

(*Received* 16 *June*, 1964)

A normal distribution with varying standard deviation has been considered in this note. Statistical properties of the distribution and the estimation of the parameters involved therein are also discussed.

There are some practical situations in which the observed variables may have normal distribution with parameters varying from observation to observation. Bhattacharjee *et al.*[1] considered the case with uniform shift in the mean and Teichroew[2] treated the problem when the standard deviation follows a gamma distribution. In practice, however, the variations in the standard deviation may be restricted within finite limits. A case in which the standard deviation is uniformly distributed over the range $[\alpha, \beta]$ is considered in this note. Statistical properties of the distribution and the estimation of the parameters are also discussed.

DERIVATION OF THE DISTRIBUTION AND STATISTICAL PROPERTIES

Without loss of generality, the conditional density of x given σ is assumed to be

$$f\left(x/\sigma\right) = \frac{1}{\sqrt{2\pi}\ \sigma}\, e^{-x^2/2\sigma^2} \qquad \left(-\infty < x < \infty\right) \quad (\sigma > 0) \tag{1}$$

where σ has a probability density

$$\left.\begin{aligned} f(\sigma) &= 1/(\beta-\alpha) \text{ if } \alpha < \sigma < \beta \\ &= 0 \text{ otherwise, where } \alpha > 0 \end{aligned}\right\} \tag{2}$$

Multiplying these probability densities and integrating over the range involved, we obtain for the marginal density :

$$f(x) = \frac{1}{(\beta-\alpha)\, 2\sqrt{2\pi}} \left\{ e^{-x^2/2\beta^2}\, \psi\left[1;\, 1;\, \frac{x^2}{2\beta^2}\right] - e^{-x^2/2\alpha^2}\, \Psi\left[1;\, 1;\, \frac{x^2}{2\alpha^2}\right] \right\} \quad (-\infty < x < \infty) \quad (\beta > \alpha > 0) \tag{3}$$

using the result

$$\int_x^\infty e^{-t}\, t^{-1}\, dt = e^{-x}\, \psi\,[1;\, 1;\, x],$$

where $\psi\,[a;\ b;\ x]$ is the well-known function due to Tricomi[3] defined for $R_e\ a > 0$. Let us designate by $f\,(x;\ \sigma,\ \delta)$ the distribution obtained from (3) by the substitution $\beta = \sigma + \delta$, $\alpha = \sigma - \delta$ with $\sigma > \delta > 0$. This means that the standard deviation is uniformly distributed with mean value at σ and maximum deviation δ on either side of the mean. It may be seen through an easy limit procedure that as δ approaches zero, $f\,(x;\ \sigma,\ \delta)$ tends to the probability density (1).

For the moments of $f(x; \sigma, \delta)$, it is evident that $\mu'_{2k+1} = 0$ $(k=0, 1, 2, \ldots\ldots)$ so that in particular the mean is zero and so all are odd moments. For even order moments, we have

$$\mu'_{2k} = \mu_{2k} = \int_{-\infty}^{\infty} x^{2k} \; f(x; \sigma, \delta) \; dx$$

$$K = \frac{1\cdot3\cdot5\ldots\ldots(2k-1)}{2\;(2k+1)\;\delta} \left[(\sigma+\delta)^{2k+1} - (\sigma-\delta)^{2k+1} \right]$$

$$k = 1, 2, \ldots\ldots \qquad (4)$$

by using the following result for the Laplace transform[4] of Ψ $[a; b; x]$:

$$\int_o^{\infty} e^{-st} \; t^{b-1} \; \Psi[a; c; t]\, dt = \frac{\Gamma(b) \; \Gamma(1+b-c)}{\Gamma(1+a+b-c)} \; {}_2F_1[b, 1+b-c; 1+a+b-c; \; 1-s],$$

for $R_e \; b>o$, $R_e \; c< \; R_e \; b+1$ and $| \; 1-S \; | < 1$

where ${}_2F_1$ $[a, b; c; x]$ is the well-known Gauss function defined by

$${}_2F_1\;[a, b; c; x] = \sum_{n=o}^{\infty} \frac{(a)_n \;\; (b)_n}{(c)_n} \cdot \frac{x^n}{n!},$$

the series being absolutely convergent whenever $| \; x \; | < 1$ and when $x=1$, provided that R_e $(c-a-b) > 0$ and when $x=-1$, provided that R_e $(c-a-b) > -1$, where

$$(a)_n = a(a+1)\ldots\ldots(a+n-1) \text{ with}$$
$$(a)_o = 1.$$

In particular

$$\mu_2 = \sigma^2$$
$$\mu_4 = 3\sigma^4 + 6\sigma^2\,\delta^2 + \tfrac{3}{5}\,\delta^4$$
$$\mu_6 = 15 \left(\sigma^6 + 5\sigma^4\;\delta^2 + 3\sigma^2\delta^4 + \frac{\delta^6}{7}\right)$$
$$\mu_8 = 35 \left(3\sigma^8 + 28\sigma^6\delta^2 + 42\sigma^4\delta^4 + 12\sigma^2\delta^6 + \frac{\delta^8}{3}\right)$$

which we shall use later.

ESTIMATION OF THE PARAMETERS

Let $(r_1, x_2, \ldots\ldots\ldots, x_n)$ be a random sample from the distribution specified by the probability density $f(x\,; \sigma, \delta\,)$. The moment estimators for σ^2 and δ^2, provided they exist, are given by

$$\hat{\sigma}^2 = \nu_2 \qquad (6)$$

$$\hat{\delta}^2 = 5 \left[-\nu_2 + \sqrt{\left(\frac{4}{5}\;\nu_2^{\,2} + \frac{\nu_4}{15}\right)} \right],$$

where

$$\nu_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r$$

Evidently, the large sample variance of $\hat{\sigma}^2$ is $\frac{1}{n}\left[2\sigma^4 + 6\sigma^2\delta^2 + \frac{3}{5}\sigma^4\right]$

and that for $\hat{\delta}^2$ $(\equiv\hat{\lambda})$ is obtained by using Cramer's formula[5]

$$V(\hat{\lambda}) = \left(\frac{\partial\hat{\lambda}}{\partial\nu_2}\right)^2 \mu_2'(\nu_2) + 2\left(\frac{\partial\hat{\lambda}}{\partial\nu_2}\right)\left(\frac{\hat{\lambda}}{\partial\nu_4}\right)\mu_{11}'(\nu_2,\nu_4) + \left(\frac{\partial\hat{\lambda}}{\partial\nu_4}\right)^2 \mu_2'(\nu_4) \quad (7)$$

where the partial derivatives are to be evaluated at the points $\nu_2 = \mu_2'$ and $\nu_4 = \mu_4'$ and the second order moments for the sample moments are obtained by using (5) and the formula given in Kendall[6] as given below:

$$\left.\begin{aligned}
\mu_2'(\nu_2) &= \frac{1}{n}\left[2\sigma^4 + 6\sigma^2\delta^2 + \frac{3}{5}\delta^4\right] \\
\mu'_2(\nu_4) &= \frac{1}{n}\left[96\sigma^8 + 944\sigma^6\delta^2 + \frac{7152}{5}\sigma^4\delta^4 + \frac{2064}{5}\sigma^2\delta^6 + \frac{848}{75}\delta^8\right] \\
\mu'_{11}(\nu_2,\nu_4) &= \frac{1}{n}\left[12\sigma^6 + 69\sigma^4\delta^2 + \frac{222}{5}\sigma^2\delta^4 + \frac{15}{7}\delta^6\right]
\end{aligned}\right\} \quad (8)$$

The partial derivatives are found to be

$$\frac{\hat{\lambda}}{\partial\nu_2} = \frac{-5(\sigma^2+\delta^2)}{(5\sigma^2+\delta^2)}, \quad \frac{\partial\hat{\lambda}}{\partial\nu_4} = \frac{5}{6(5\sigma^2+\delta^2)} \quad (9$$

Using then (7), (8) and (9), we obtain, for the large sample variance of $\hat{\delta}^2$, the following expression.

$$V\left(\hat{\delta}^2\right) = \frac{25}{36n(5\sigma^2+\delta^2)^2}\left\{24\sigma^8 + 332\sigma^6\delta^2 + \frac{2976}{5}\sigma^4\delta^4 + \frac{3972}{35}\sigma^2\delta^6 + \frac{3776}{525}\delta^8\right\} \quad (10)$$

ACKNOWLEDGEMENTS

We are grateful to Dr. P. V. Krishna Iyer and Dr. N. K. Chakravarti for their keen interest and encouragement. We also wish to acknowledge our thanks to Dr. A. M. Kshirsagar, Dr. N. K. Jaiswal and Mr. D. Ray for their help.

REFERENCES

1 BHATTACHARJEE, G.P., Pandit, S.N.N. & Mohan, R., *Technometrics*, **5** (1963), 404.

2 TEICHROEW, D., *Ann. Math. Statist.*, **28** (1957), 510.

3 ERDELYI, ARTHUR, *et al.*, "Higher Transcendental functions Vol. I" (McGraw Hill Book Company Inc. New York), 1953.

4 SLATER, L.J., "Confluent Hypergeometric functions" (Cambridge University Press, Cambridge), 1960.

5 CRAMER, H., "Mathematical Methods of Statistics" (Princeton University Press, Princeton, USA), 1946.

6 KENDALL, M.G., "The Advanced theory of Statistics, Vol. I" (Charles Griffin & Co., London), 1948.