BULK QUEUES WITH ARBITRARY ARRIVALS AND EXPONENTIAL SERVICE TIME DISTRIBUTIONS

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The bulk queuing problem has been studied under the assumption that the input is restricted to a wide class of arrival time distributions and exponential service. Both time dependent and steady state cases have been examined. The results have been specialised for single arrivals and Erlangian inputs.

The steady state solution of the bulk service queuing problem was first obtained by Bailey[1] by using the 'Imbedded Markov chain' technique and later by Jaiswal[2] by the phase technique. Downtown[3] gave the solution for the waiting time distribution. Jaiswal[4] obtained the time dependent solution utilising the phase technique and also the busy period distribution[5] for the time dependent case. In all these cases a Poisson input and a general service time distribution were assumed, the units arriving singly but being serviced in groups of S or less. Miller[6] studied a generalised model of queues where units arrive and are served in batches. Recently Keilson[7] has studied the bulk queuing process with random arrival epochs and arbitrary service times, considering the general bulk queue as a Hilbert problem.

Two types of bulk service queuing processes have been considered: (*i*) the 'transportation' type of bulk queues considered by Bailey and Downtown where the server on finding an empty queue does not wait and the next arriving unit has to wait for the next service epoch; (*ii*) the ordinary bulk service queuing process, considered by Jaiswal and Miller where, as in the classical queuing process, the server on finding an empty queue waits for a unit to arrive to start the next service.

The two processes are quite different, but in the steady state, Jaiswal[4] has shown that the probability that there are n units just before a service is to commence, is the same for both processes with Poisson inputs and arbitrary service time distributions.

In this paper a bulk queuing process of the transportation type, with an arbitrary arrival time distribution, has been considered with the help of the phase method. The study also incorporates the idea of batch arrivals and batch services discussed by Miller[6]. The results have been specialised for single arrivals and Erlangian inputs.

Units arrive at a service station in batches, the size of the batch being a random variable, *i.e.* the size of an arrival batch is N with probability b_N $(\Sigma\, b_N = 1)$. The inter-arrival time between successive batches are independent random variables and the arrival pattern of the batches is described as follows:

We assume the existence of an arrival-timing channel having an arbitrary number of phases, the time of staying in any of these phases being identically, independently and exponentially distributed with mean $1/\lambda$. A reservoir of infinite capacity attached to this channel emits a group of units when it finds no one in the channel. The emitted group goes to the rth phase with probability C_r. After staying in that phase for a certain time, the group moves to the $(r-1)$th phase, and then to the $(r-2)$th and so on, and finally from the first phase into the queue. When the arrival-time channel thus becomes empty, another group of units is emitted into the channel from the reservoir and the process is repeated. As usual the phases are labelled in the reverse order. This method of simulating an arrival time or service time distribution is well established and the method of choosing the C_r and λ, as given by Luchak[8], makes this method an elegant, but approximate, technique to generate a wide class of arrival distributions met in practice.

It is seen from the above that inter-arrival times are identically and independently distributed according to

$$A(t)\,dt = \sum_{r=1}^{j} C_r \frac{(\lambda t)^{r-1}}{(r-1)!} e^{-\lambda t} \lambda dt$$

The service mechanism is described thus—

A service station offers service at certain epochs of time in such a way that at each service epoch the server decides to take a batch of size M with probability d_M ($\Sigma\, d_M = 1$) and accepts for service M units or the whole length of the queue whichever is less. The inter-service times are random variables having a negative exponential distribution with mean $1/\mu$. Miller[6] further distinguishes between Model I and Model II depending on whether late arrivals join the batch in service or not, in case the 'quota' for the service batch has not been met. However, this distinction does not arise in a transportation type bulk queuing process as any new arrival shall have to wait for the next service epoch.

Further we assume that there exist two numbers m_o and n_o such that $b_N = o$ for $N > n_o$ and $d_M = o$ for $M > m_o$.

TIME—DEPENDENT SOLUTION

Let $P_{n,r}(t)$ denote the probability that at time t there are n units in the queue waiting for service and the arrival group is in the r-th phase of the arrival timing channel. Then the following set of different—differential equations describes the process.

$$P'_{n,r}(t) = -(\lambda + \mu) P_{n,r}(t) + \lambda P_{n,r+1}(t) + \lambda C_r \sum_{N=1}^{n_o} b_N \; P_{n-N,1}(t) + \mu \sum_{M=1}^{m_o} d_M P_{n+M,r}(t)$$

$$(n > o,\; r < j) \qquad (1)$$

$$P'_{n,j}(t) = -(\lambda + \mu) P_{n,j}(t) + \lambda C_j \sum_{N=1}^{n_o} b_N P_{n-N,1}(t) + \mu \sum_{M=1}^{m_o} d_M \; P_{n+M,j}(t) \quad (n > o) \qquad (2)$$

$$P'_{o,r}(t) = -\lambda P_{o,r}(t) + \lambda P_{o,r+1}(t) + \mu \sum_{M=1}^{m_o} d_M \sum_{n=1}^{M} P_{n,r}(t) \qquad (r < j) \qquad (3)$$

$$P'_{o,j}(t) = -\lambda P_{o,j}(t) + \mu \sum_{M=1}^{m_o} d_M \sum_{n=1}^{M} P_{n,j}(t) \qquad (4)$$

Let us define the Laplace transform of $P_{n,r}(t)$ as

$$P_{n,r}(a) = \int_{o}^{\infty} e^{-at} P_{n,r}(t)\,dt \qquad (R_e\; a > o) \qquad (5)$$

Let us also assume that the system starts with no units at time $t = 0$.

i.e., $\quad P_{n,r}(o) = o \qquad$ *for* $\quad n > o$

$$P_{o,r}(o) = C_r \qquad (r = 1, 2, \ldots\ldots j)$$

Taking the Laplace transform of the above set of equations (1) to (4) we obtain

$$-(\lambda + \mu + \alpha)\, P_{n,r}(\alpha) + \lambda\, P_{n,r+1}(\alpha) + \lambda\, C_r \sum_{N=1}^{n_o} b_N\, P_{n-N,1}(\alpha)$$

$$+ \mu \sum_{M=1}^{m_o} d_M \quad P_{n+M,r}(\alpha) = o \qquad (n > o,\ r < j) \qquad (6)$$

$$-(\lambda + \mu + \alpha)\, P_{n,j}(\alpha) + \lambda\, C_j \sum_{N=1}^{n_o} b_N\, P_{n-N,1}(\alpha) + \mu \sum_{M=1}^{m_o} d_M \; P_{n+M,j}(\alpha) = o$$

$$(n > o) \qquad (7)$$

$$-(\lambda + \alpha)\, P_{o,r}(\alpha) + \lambda\, P_{o,r}(\alpha) + \lambda\, P_{o,r+1}(\alpha) + \mu \sum_{M=1}^{m_o} d_M \sum_{n=1}^{M} P_{n,r}(\alpha) + C_r = o$$

$$(r < j) \qquad (8)$$

$$-(\lambda + \alpha)\, P_{o,j}(\alpha) + \mu \sum_{M=1}^{m_o} d_M \sum_{n=1}^{M} P_{n,j}(\alpha) + C_j = o \qquad (9)$$

Let us define the following generating functions :

$$Q_n(x\,;\alpha) = \sum_{r=1}^{j} x^r\, P_{n,r}(\alpha)$$

$$F(x, y; \alpha) = \sum_{n=o}^{\infty} y^n\, Q_n(x; \alpha)$$

Multiplying the set of equations (6) to (9) by appropriate powers of x and y and summing up over n and r we get,

$$\left\{-(\lambda + \mu + \alpha) + \frac{\lambda}{x} + \mu\, D(y)\right\} F(x, y; \alpha)$$

$$-\lambda \left\{ 1 - B(y)\, C(x) \right\} \sum_{n=o}^{\infty} y^n P_{n,1}(\alpha)$$

$$+ \mu \sum_{r=1}^{j} x^r \sum_{M=1}^{m_o} \sum_{n=o}^{M-1} \frac{d_M}{y^M} \left(y^M - y^n \right)_{n,r}(\alpha) + C(x) = o \qquad (10)$$

Where

$$C(x) = \sum_{r=1}^{j} C_r\, x^r$$

$$B(y) = \sum_{N=1}^{n_o} b_N\, y^N$$

$$\text{and} \quad D(y) = \sum_{M=1}^{m_o} (d_M / y^M)$$

Choosing x, such that the coefficient of $F(x, y; \alpha)$ in (10) becomes zero, helps us in evaluating $\sum_{n=o}^{\infty} y^n\, P_{n,1}(\alpha)$. Accordingly, if we put

$$x = \frac{\lambda}{(\lambda + \mu + \alpha) - \mu D(y)} = Y \text{ (say), in (10)}$$

and simplify, we get

$$G(y;\, \alpha) = \sum_{n=o}^{\infty} y^n P_{n,1}(\alpha)$$

$$= \frac{\mu \sum\limits_{r=1}^{j} Y^r \sum\limits_{M=1}^{m_\circ} \sum\limits_{n=o}^{M-1} \frac{d_M}{y^M} (y^M - y^n) P_{n,r}(\alpha) + C(Y)}{\lambda \{ 1 - B(y)\, C(Y) \}} \qquad (11)$$

Substituting this value of $G(y;\, \alpha)$ in (10) and putting $x = 1$ we obtain

$$F(1, y;\, \alpha) = \sum_{n=o}^{\infty} y^n Q_n(1;\, \alpha)$$

$$= \frac{\mu \sum\limits_{r=1}^{j} \sum\limits_{M=1}^{m_\circ} \sum\limits_{n=0}^{M-1} \frac{d_M}{y^M} (y^M - y^n) P_{n,r}(\alpha) - \lambda \{1 - B(y)\} G(y;\alpha) + 1}{\mu \{ 1 - D(y) \} + \alpha} \qquad (12)$$

Where $Q_n(1, \alpha) = \sum\limits_{r=1}^{j} P_{n,r}(\alpha)$ is the Laplace transform of the probability that there are n units in the queue at time t. The generating function (12) of the Laplace transform of the queue length probabilities is uniquely determined if the $m_\circ j$ unknown probabilities $P_{n,r}(\alpha)$, ($n = o, 1, 2, .. m_\circ^{-1}$; $r = 1, .. j$) in the numerator of $G(y;\, \alpha)$, can be determined. For this purpose we shall make use of the fact that $G(y;\, \alpha)$ is convergent inside the unit circle $|y| = 1$. By using Rouches theorem it can be shown that the denominator of (11) has $m_\circ j$ zeros inside and n_o zeros outside the unit circle. For $G(y;\, \alpha)$ to be regular inside $|y| = 1$, the zeros of the numerator and denominator of (11) must coincide at least inside $|y| = 1$, *i.e.*, the numerator must vanish for the zeros of modulus less than unity of the denominator. This condition gives rise to a set of $m_\circ j$ linear equations

$$\mu \sum_{r=1}^{j} Y_i^r \sum_{M=1}^{m_\circ} \sum_{n=o}^{M-1} \frac{d_M}{y^M} (y_i^M - y_i^n) P_{n,r}(\alpha) + C(Y_i) = o$$

$$(i = 1, 2, \ldots m_\circ j) \qquad (13)$$

Where y_i are the roots of modulus less than unity of the equation $1 - B(y)\, C(Y) = o$ and $Y_i = \lambda / \{ (\lambda + \mu + \alpha) - \mu D(y_i) \}$. The set of equations (13) will uniquely determine $G(y;\, \alpha)$ and hence $F(1, y;\, \alpha)$ provided the $m_\circ j$ equations are linearly independent. We proceed to show that this condition is satisfied for the case of single arrivals and fixed batch size for service and try to specialise the results for Erlangian arrivals.

Single arrivals and fixed batch size for service—In this case, we have $b_1 = 1$ and $b_N = o$ for $N \neq 1$ and $d_M = 1$ for $M = S$ (say)
$= o$ for $M \neq S$. (11) reduces to

$$G(y;\, \alpha) = \frac{\frac{\mu}{y^S} \sum\limits_{m=o}^{S-1} \sum\limits_{r=1}^{j} Y^r (y^S - y^m) P_{m,r}(\alpha) + C(Y)}{\lambda \{ 1 - y\, C(Y) \}} \qquad (14)$$

where $Y = \lambda y^S / \{ (\lambda + \mu + \alpha) y^S - \mu \}$, and the set of equations (13) reduces to

$$\mu \sum_{m=o}^{S-1} \sum_{r=1}^{j} Y_i^r \; (y_i^S - y_i^m) \; P_{m,r} \; (\alpha) + y_i^S \, C \, (Y_i) = o$$

$$(i = 1, 2, \ldots . \; Sj) \qquad (15)$$

where y_i are the roots of modulus less than unity of the equation $1 - y\,C(y) = 0$. It can be shown in this case that the Sj equations are linearly independent if we assume all the Sj zeros of modulus less than unity of the denominator are simple for, the determinant of the coefficients of $P_{m,r}\;(\alpha)$ in (15) can be shown to reduce to

$$\Delta = \prod_{i=1}^{Sj} \left[\frac{y_i^S \, (y_i - 1)}{\{ (\lambda + \mu + \alpha) \, y_i^S - \mu \}^j} \right] (\lambda \;\; \mu)^{\frac{S}{2} j (j-1) + 1} \begin{vmatrix} 1 & y_1 & \ldots . & y_1^{Sj-1} \\ 1 & y_2 & \ldots . & y_2^{Sj-1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1 & y_{Sj} & \ldots . & y_{Sj}^{Sj-1} \end{vmatrix}$$

$$= (-1)^{\frac{S(S+1)j(j-1)}{4}} \; (\lambda \, \mu)^{\frac{S}{2} j(j-1)+1} \prod_{i=1}^{Sj} \frac{y_i^S \, (y_i - 1)}{\{ (\lambda + \mu + \alpha) \, y_i^S - \mu \}^j} \prod_{\substack{i=1 \\ i<j}}^{Sj} (y_i - y_j)$$

It will be seen from this that $\triangle$ will be nonvanishing and therefore the Sj equations are linearly independent provided $y_i \neq$ o, 1 or $y_i \neq y_j$ for all i and j. It will be shown below that in the case of Erlang arrivals all these conditions are fulfilled.

The generating function of the Laplace transform of the queue length probabilities is given by (12), when $b_1 = 1$ and $d_S = 1$ are substituted, as

$$F \; (1, \; y; \; \alpha) = \frac{\mu \sum_{m=o}^{S-1} \sum_{r=1}^{j} (y^S - y^m) \, P_{m,r} \; (\alpha) + y^S - \lambda \, y^S \, (1 - y) \, G \, (y; \alpha)}{(\mu + \alpha) \, y^S - \mu} \qquad (16)$$

The Laplace transform of the mean number of units in the queue at time t is given by

$$\frac{\partial}{\partial y} F \, (1, y; \alpha) \Big|_{y=1} = \frac{1}{\alpha} \left\{ \mu \sum_{m=o}^{S-1} \sum_{r=1}^{j} (s - m) \, P_{m,r} \; (\alpha) + \lambda \, G \, (1, \alpha) \right\} - S\mu/\alpha^2 \qquad (17)$$

Particular case—Let the arrival time distribution be k—Erlang *i.e.*, $C_r = 1$ for $r = k$ and $C_r = 0$ for $r \neq k$.

Then we have

$$G \, (y; \alpha) = \frac{\mu/y^S \times \sum_{m=o}^{S-1} \sum_{r=1}^{k} Y^r \left(y^S - y^m \right) P_{m,r} \; (\alpha) + Y^k}{\lambda \left\{ 1 - y \, Y^k \right\}} \qquad (18)$$

where

$$Y = \lambda y^S \, / \, \{ (\lambda + \mu + \alpha) \, y^S - \mu \}$$

We shall show that $G(y\,; \alpha)$ can be uniquely determined in this case. First it will be shown that the Sk zeros of the denominator are simple provided $| \; \alpha \; | > \lambda - \mu Sk$. For, the zeros of the denominator will be simple if the equations $\lambda^k y^{Sk+1} = \{(\lambda+\mu+\alpha) y^S - \mu\}^k$ and $(Sk+1)\lambda^k y^{Sk} = kS(\lambda+\mu+\alpha)y^{S-1}\{(\lambda+\mu+\alpha)y^S - \mu\}^{k-1}$, are not satisfied by any y whose modulus is less than unity. Dividing and simplifying we get $y^S = \frac{\mu(Sk+1)}{\lambda+\mu+\alpha}$, the possible repeated zero. Therefore, if $\left|\frac{\mu(Sk+1)}{\lambda+\mu+\alpha}\right| > 1$ there will be no multiple zeros inside and on the unit circle. Now $\mu(Sk+1) > |\lambda+\mu+\alpha|$ and since $|\lambda+\mu+\alpha| \geqslant |\lambda+\mu| - |\alpha|$, $\mu(Sk+1) > \lambda+\mu - |\alpha|$ *i.e.*, $|\alpha| > \lambda - \mu Sk$. Therefore if α is chosen to satisfy this condition, which is always possible, since we require $R_e(\alpha) > 0$, the zeros inside the unit circle will be simple.

Now, the Sk equations to evaluate the $P_{m,r}(\alpha)$ in the numerator of $G(y;\alpha)$ are

$$\mu \sum_{m=0}^{S-1} \sum_{r=1}^{k} Y_i^{r} (y_i^{S} - y_i^{m}) P_{m,r}(\alpha) + y_i^{S} Y_i^{k} = 0 \quad (i = 1, 2 \ldots\ldots Sk) \qquad (19)$$

where y_i are the Sk simple zeros of the denominator which lie inside the unit circle. These equations will be linearly independent if the determinant $\triangle$ of the coefficients of $P_{m,r}(\alpha)$ in (19) is nonvanishing. It can be shown that $\triangle$ will reduce to

$$(-1)^{\frac{S(S+1)k(k-1)}{4}} (\lambda\mu)^{\frac{S}{2}k(k-1)+1} \prod_{i=1}^{Sk} \frac{y_i^{S}(y_i-1)}{\{(\lambda+\mu+\alpha)y_i^{S} - \mu\}^{k}} \prod_{\substack{i=1\\ i<j}}^{Sk} (y_i - y_j)$$

since $y_i \neq 0$ or 1 and y_i is different, $\triangle \neq 0$, and hence $P_{m,r}(\alpha)$ $[m = 0, 1, 2, \ldots S-1 : r = 1, 2, \ldots k]$ can be uniquely determined.

The explicit solutions, by inverting the generating function, is difficult to obtain, and hence let us restrict ourselves to the simplest case in which the units arrive according to a Poisson distribution, *i.e.*, we take $k=1$. In this case $G(y;\alpha) = F(1,y;\alpha) = C(\alpha)/(\hat{y}_o - y)$ where $\hat{y}_o$ is the root of modulus greater than unity of the equation $\lambda y^{S+1} - (\lambda+\mu+\alpha) y^S + \mu = 0$ and $C(\alpha)$ is to be determined. By putting $y = 1$, it is seen that $C(\alpha) = (\hat{y}_o - 1)/\alpha$ and hence $Q_n(1, \alpha)$, the Laplace transform of the probability that there are n units in the queue, is given by $\frac{(\hat{y}_o - 1)}{\alpha} \hat{y}_o^{-(n+1)}$

In this case explicit expressions for the time dependent probabilities can be obtained by following the method outlined by Luchak and Jaiswal.

STEADY STATE SOLUTION

By supressing the time suffix t in equations (1) to (4) and equating the left hand side to zero and defining the generating functions $Q_n(x)$ and $F(x,y)$ similar to the ones defined for the time-dependent case, one can obtain the generating function of the state probabilities or they can be obtained by using the well-known property of the Laplace transform: $\lim_{t\to\infty} f(t) = \lim_{\alpha\to 0} \alpha f(\alpha)$, provided the limit on the left hand side exists. Applying this to (11) and (12) we get

$$G(y) = \sum_{n=o}^{\infty} y^n \, p_{n,1} = \frac{\mu \sum_{r=1}^{j} Y^r \sum_{M=1}^{m_o} \sum_{n=o}^{M-1} \frac{d_M}{y^M} (y^M - y^n) p_{n,r}}{\lambda \{1 - B(y)\, C(Y)\}} \tag{20}$$

and

$$F(y) = \frac{\mu \sum_{r=1}^{j} \sum_{M=1}^{m_o} \sum_{n=o}^{M-1} \frac{d_M}{y^M} (y^M - y^n) p_{n,r} - \lambda \{1 - B(y)\} G(y)}{\mu \{1 - D(y)\}} \tag{21}$$

where $Y = \lambda / \{(\lambda + \mu) - \mu D(y)\}$. The evaluation of $m_0 j$ unknown probabilities $p_{n,r}$ $(n = 0, 1 \ldots m_o - 1;\ r = 1,2 \ldots j)$ in the numerator of (20) is done by making use of the condition that $G(y)$ must be convergent inside the unit circle. By putting $y = 1/\alpha$; $b_M = p_M$ in $1 - B(y)\, C(Y)$, we get equation (21) of Miller[6], and it can be seen that this equation has $m_o j - 1$ zeros inside, one on the unit circle and n_o zeros outside it. Using these $m_o j - 1$ zeros inside the unit circle we get $m_o j - 1$ numerator equations in $m_o j$ unknowns. Further the fact that $F(1) = 1$ gives rise to one more equation. Thus we have m_{oj} equations in m_{oj} unknowns and this should suffice to uniquely determine the $p_{n,r}$ and hence $G(y)$ and $F(y)$. We shall presently see that in the special cases $G(y)$ and $F(y)$ could be uniquely determined. Before proceeding to the special cases, we would like to express $G(y)$ in terms of the zeros of the denominator of (20) that lie outside the unit circle.

Let $y_1, y_2 \ldots y_{n_o}$ be the n_o zeros of $1 - B(y)\, C(Y) = 0$ that lie outside $|y| = 1$ Then it is easily seen that

$$G(y) = A / \prod_{i=1}^{n_o} (y_i - y) \tag{22}$$

where A is a constant to be determined.

Putting $y = 1$ in (22), we get

$$G(1) = A / \prod_{i=1}^{n_o} (y_i - 1) \tag{23}$$

and therefore

$$\frac{G(y)}{G(1)} = \frac{\sum_{n=0}^{\infty} y^n \, p_{n,1}}{\sum_{n=0}^{\infty} p_{n,1}} = \prod_{i=1}^{n_o} \left(\frac{y_i - 1}{y_i - y} \right) \tag{24}$$

which defines the generating function of the stationary probabilities

$$\pi_n = p_{n,1} \Big/ \sum_{n=o}^{\infty} p_{n,1}, \quad (n = 0,1,2 \ldots),$$

π_n being the probability that there are n units in the system just before a new arrival takes place. This agrees with the stationary distribution $\{\pi_n\}$ obtained by Miller[6] for $n_o = 1,2$ and 3.

It may be mentioned here that the above analysis points out to the fact that the stationary distribution $\{\pi_n$ of the probabilities that there are n units in the system just before an arrival takes place is the same for both the ordinary and transportation type of bulk queuing processes. This is similar to the result obtained by Jaiswal[4].

Special Cases: Single arrivals and fixed batch size for service—Putting $b_1=1$ and $d_s = 1$ as we did earlier, $G(y)$ and $F(y)$ reduce to

$$G(y) = \frac{\frac{\mu}{y^S} \sum_{m=o}^{S-1} \sum_{r=1}^{j} Y^r (y^S - y^m) p_{m,r}}{\lambda \{1 - y C(Y)\}} \tag{25}$$

and

$$F(y) = \frac{\mu \sum_{m=o}^{S-1} \sum_{r=1}^{j} (y^S - y^m) p_{m,r} - \lambda y^S (1-y) G(y)}{\mu (y^S - 1)} \tag{26}$$

where $Y = \lambda y^S / \{(\lambda + \mu) y^S - \mu\}$

The $Sj-1$ zeros of modulus less than unity of $1-y\,C(Y)$, when substituted in the numerator of $G(y)$, give rise to the following set of $Sj-1$ homogeneous linear equations in Sj unknowns:

$$\sum_{m=o}^{S-1} \sum_{r=1}^{j} Y_i^r (y_i^S - y_i^m) p_{m,r} = 0 \qquad (i = 1, 2, \ldots\ldots\ldots Sj-1) \tag{27}$$

By putting $y=1$ in (26) and using the condition $F(1)=1$, we obtain

$$\sum_{m=o}^{S-1} \sum_{r=1}^{j} (S-m) p_{m,r} = S - \lambda/\mu \sum_{r=1}^{j} r\, C_r \tag{28}$$

Equation (28) along with (27) will uniquely determine the $p_{m,r}$ and hence $G(y)$ and $F(y)$ provided the above set forms a consistent set. In the case of Erlang arrivals considered below it is shown that this condition is satisfied. It may be noted that for the existence of a steady-state solution it is necessary that $S\mu \sum_{r=1}^{j} r\, C_r > \lambda$ for, otherwise (28) becomes negative.

The mean number of units in the system is given by

$$F'(1) = \frac{1}{2S} \sum_{m=o}^{S-1} \sum_{r=1}^{j} m (S-m)\, p_{m,r} + \frac{\lambda (S+1)}{2 \mu S} G(1) + \frac{\lambda}{\mu S} G'(1) \tag{29}$$

Particular case—Let us consider a k-Erlang arrival distribution *i.e.*, $C_r = 1$ for $r=k$ and $C_r = 0$ for $r \neq k$.

In this case

$$G(y) = \frac{\frac{\mu}{y^S} \sum_{m=o}^{S-1} \sum_{r=1}^{k} Y^S (y^s - y^m)\, p_{m,r}}{\lambda \{1 - y Y^k\}} \tag{30}$$

and

$$F(y) = \frac{\mu \sum_{m=o}^{S-1} \sum_{r=1}^{k} (y^S - y\ )\, p_{m,r} - \lambda y^S (1-y) G(y)}{\mu (y^S - 1)} \tag{31}$$

Proceeding on the same lines as in the time-dependent case, we can show that the denominator of (30) has no multiple roots of modulus less than unity provided $\mu S k > \lambda$.

This conidition has to be satisfied for the steady state as already pointed out. If y_i ($i = 1,2. \ldots . S_k - 1$) are the roots of modulus less than unity, then equations (27) and (28) for k-Erlang case will be consistent provided the determinant $\triangle$ of the coefficients of $p_{m,r}$ is nonvanishing. In fact, it can be shown that $\triangle$ reduces to

$$(-1)^{\frac{S(S+1)k(k-1)}{4} + Sk + 1} \lambda^{1-k} \prod_{i=1}^{Sk-1} \frac{y_i^{S}(y_i - 1)^2}{\{(\lambda + \mu)y_i^{S} - \mu\}^k} \prod_{\substack{i=1 \\ i<j}}^{Sk-1} (y_i - y_j)$$

since $y_i \neq 0$ or 1 and since no two y_j are equal, $\triangle$ is nonvanishing and hence the $p_{m,r}$'s can be uniquely determined.

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