SOME INTEGRALS INVOLVING THREE BESSEL FUNCTIONS

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In this paper we shall evaluate the integrals of the following type:—

$$I = \int_0^1 XJ_0(\lambda X)I_s(\mu X)I_s(\nu X)dX$$

$$J = \int_0^1 XJ_0(\lambda X)J_s(\mu X)J_s(\nu X)dX$$

$$K = \int_0^1 X^\lambda J_\nu(bX)J_0(cX)J_0(dX)dX$$

which we have come across in the solution of some non-linear problems with central or axial symmetry.

Evidently these integrals provide Fourier-Bessel expansions for $I_s(\mu X)I_s(\nu X)$, $J_s(\mu X)J_s(\nu X)$ and $J\nu(bX)J_0(cX)$ respectively if λ is one of the positive zeroes of the function $J_0(X)$. To the best knowledge of the author the values of these integrals are not available in the current literature.

1. Here we shall evaluate the following integral

$$I = \int_0^1 XJ_0(\lambda X)I_s(\mu X)I_s(\nu X)dX \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

We shall use the following results:—

$$\int_0^1 XJ_0(\lambda X)J_0(KX)dX = \frac{\lambda}{K^2 - \lambda^2} J_0'(\lambda)J_0(K) \quad \ldots \quad \ldots \quad (1\cdot1)$$

and the addition theorem

$$J_0\left\{X\sqrt{\mu^2 + \nu^2 - 2\mu\nu\cos\phi}\right\} = \sum_{s=0}^{\infty} \epsilon_s J_s(\mu X)J_s(\nu X)\cos S\phi \quad \ldots \quad (1\cdot2)$$

where $\epsilon_0 = 1$, $\epsilon_s = 2$ for $S = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Taking $K = i\sqrt{\mu^2+\nu^2-2\mu\nu\cos\phi}$ in (1·1) and integrating with respect to ϕ between o and π after multiplying by $\cos S\phi$ we get

$$\int_o^1 X J_o(\lambda X) I_s(\mu X) I_s(\nu X)\,dX$$

$$= \frac{(-1)^s \lambda J_1(\lambda)}{\pi\epsilon_s}\sum_{m=o}^{\infty}(-1)^m \epsilon_m I_m(\mu) I_m(\nu)\int_o^{\pi}\frac{\cos(m+s)\phi+\cos(m\sim s)\phi}{\lambda^2+\mu^2+\nu^2-2\mu\nu\cos\phi}\,d\phi$$

$$= \frac{(-1)^s \lambda J_1(\lambda)}{\epsilon_s}\sum_{m=o}^{\infty}\frac{(-1)^m \epsilon_m I_m(\mu) I_m(\nu)}{(a^2-b^2)^{1/2}}\left[\left\{\frac{-\sqrt{a^2-b^2}+a}{b}\right\}^{m+s}\right.$$

$$\left.+\left\{\frac{-\sqrt{a^2-b^2}+a}{b}\right\}^{m\sim s}\right] \qquad (1\cdot3)$$

where $a=\lambda^2+\mu^2+\nu^2$, $b=2\mu\nu$ .. .. .. (1·4)
when $S=0$ we have the particular case

$$\int_o^1 X J_o(\lambda X) I_o(\mu X) I_o(\nu X)\,dX$$

$$= 2\lambda J_1(\lambda)\sum_{m=o}^{\infty}\frac{(-1)^m \epsilon_m I_m(\mu) I_m(\nu)}{[(\lambda^2+\mu^2+\nu^2)^2-4\mu^2\nu^2]^{1/2}}$$

$$\times\left\{\frac{\lambda^2+\mu^2+\nu^2-\sqrt{(\lambda^2+\mu^2+\nu^2)^2-4\mu^2\nu^2}}{2\mu\nu}\right\}^m \qquad (1\cdot5)$$

2. We shall now evaluate the integral

$$J = \int_o^1 X J_o(\lambda X) J_s(\mu X) J_s(\nu X)\,dX \quad .. \quad (2\cdot1)$$

We know that

$$I_\nu(z) = e^{-\frac{1}{2}\pi i\nu} J_\nu\left(z e^{\frac{1}{2}\pi i}\right) \quad .. \quad .. \quad (2\cdot2)$$

where $-\pi < \arg Z \leqslant \frac{1}{2}\pi$

or $$I_\nu\left(z e^{-\frac{1}{2}\pi i}\right) = e^{-\frac{1}{2}\pi i\nu} J_\nu(z)$$

Hence by replacing μ by $\mu e^{-\frac{1}{2}\pi i}$ and ν by $\nu e^{-\frac{1}{2}\pi i}$ we get

$$\int_o^1 X J_o(\lambda X) J_s(\mu X) J_s(\nu X)\,dX = \frac{\lambda J_o'(\lambda)}{\epsilon_s}\sum_{m=o}^{\infty}\epsilon_m J_m(\mu) J_m(\nu)\, B \quad .. \quad (2\cdot3)$$

where

$$B = \frac{1}{\pi}\int_{o}^{\pi} \frac{\cos (m+s)\phi + \cos (m \sim s)\phi}{\mu^2+\nu^2-\lambda^2-2\mu\nu\cos\phi}\, d\phi \quad .. \quad .. \quad (2\cdot4)$$

The value of B depends on the relative values of λ, μ and ν.

when $\mu \sim \nu > \lambda$,

$$B = \frac{1}{(a^2-b^2)^{\frac{1}{2}}}\left[\left\{\frac{a-\sqrt{a^2-b^2}}{b}\right\}^{m+s} + \left\{\frac{a-\sqrt{a^2-b^2}}{b}\right\}^{m\sim s}\right] \quad (2\cdot5)$$

where

$$a = \mu^2+\nu^2-\lambda^2 > o \quad , \quad b = 2\mu\,\nu$$

when $\mu + v < \lambda$

$$B = -\frac{1}{(\alpha^2-b^2)^{\frac{1}{2}}}\left[\left\{\frac{\sqrt{\alpha^2-b^2}-\alpha}{b}\right\}^{m+s} + \left\{\frac{\sqrt{\alpha^2-b^2}-\alpha}{b}\right\}^{m\sim s}\right] \quad (2\cdot6)$$

where $\alpha = \lambda^2 - \mu^2 - \nu^2 > o$

when $\mu \sim \nu < \lambda \sqrt{\mu^2+\nu^2}$ and $\sqrt{\mu^2+\nu^2} < \lambda < \mu+\nu$ the integral becomes unbounded in the range of integration. In these two cases we shall take the principal values which exist.

when $\mu \sim \nu < \lambda < \sqrt{\mu^2+\nu^2}$

$$B = \frac{1}{2\,\pi\,\mu\,\nu}\int_{o}^{\pi} \frac{\cos (m+s)\,\phi + \cos (m\sim s)\phi}{\cos\psi - \cos\phi}\, d\,\phi$$

$$= -\frac{1}{2\,\mu\,\nu}\left[\frac{\sin (m+s)\,\psi + \sin (m \sim s)\,\psi}{\sin\psi}\right] \quad .. \quad (2\cdot7)$$

where

$$0 < \cos\psi = \frac{\mu^2+\nu^2-\lambda^2}{2\,\mu\,\nu} < 1 \quad .. \quad .. \quad .. \quad (2\cdot8)$$

so that $0 < \psi < \frac{\pi}{2}$

when $\sqrt{\mu^2+\nu^2} < \lambda < \mu+\nu$

$$B = -\frac{1}{2\,\pi\,\mu\,\nu}\int_{o}^{\pi} \frac{\cos (m+s)\,\phi + \cos (m\sim s)\phi}{\cos\phi - \cos\psi}\, d\phi$$

$$= -\frac{1}{2\,\mu\,\nu}\left[\frac{\sin (m+s)\,\psi + \sin (m \sim s)\,\psi}{\sin\psi}\right] \quad .. \quad .. \quad (2\cdot9)$$

where $-1 < \cos\psi = \dfrac{\mu^2+\nu^2-\lambda^2}{2\,\mu\,\nu} < 0$

so that $\frac{\pi}{2} < \psi < \pi$

In particular when $S = O$

$$\int_o^1 X J_o(\lambda X) J_o(\mu X) J_o(\nu X)\, dX = \lambda J_o'(\lambda) \sum_{m=o}^{\infty} \epsilon_m J_m(\mu) J_m(\nu) B \tag{2·10}$$

Where

$$\left.\begin{aligned} B &= \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \left\{\frac{a-\sqrt{a^2-b^2}}{b}\right\}^m, \mu \sim \nu < \lambda \\ &= -\frac{2}{(a^2-b^2)^{\frac{1}{2}}} \left\{\frac{\sqrt{a^2-b^2}-a}{b}\right\}^m, \lambda < \mu+\nu \\ &= -\frac{1}{\mu\nu}\frac{\sin m\psi}{\sin\psi}, 0 < \psi < \frac{\pi}{2}, \mu \sim \nu < \lambda < \sqrt{\mu^2+\nu^2} \\ &= -\frac{1}{\mu\nu}\frac{\sin m\psi}{\sin\psi}, \frac{\pi}{2} < \psi < \pi, \sqrt{\mu^2+\nu^2} < \lambda < \mu+\nu \end{aligned}\right\} \tag{2·11}$$

The integrals (1·5) and (2·11) provide Fourier-Bessel expansions for $I_o(\mu X) I_o(\nu X)$ and $J_o(\mu X) J_o(\nu X)$ respectively in terms of $J_o(\lambda_n X)$ where λ's are positive zeroes of $J_o(X)$.

3. In this section we shall evaluate the integral

$$K = \int_o^1 X^\lambda J_\nu(bX) J_o(cX) J_o(dX)\, dX \quad .. \quad .. \tag{3·1}$$

From the multiplication formula (1)

$$J_\mu(aX) J_\nu(bX) = \frac{(\frac{1}{2}az)^\mu (\frac{1}{2}bz)^\nu}{\Gamma(\nu+1)} \cdot \sum_{m=o}^{\infty} (-1)^m \frac{(\frac{1}{2}az)^{2m} {}_2F_1(-m, -\mu-m; \nu+1; b^2/a^2)}{\lfloor m \; \Gamma(\mu+m+1)} \tag{3·2}$$

We have

$$\int_o^1 X^\lambda J_\mu(aX) J_\nu(bX)\, dX = \frac{(\frac{1}{2}a)^\mu (\frac{1}{2}b)^\nu}{\Gamma(\nu+1)} \sum_{m=o}^{\infty} (-1)^m \frac{(\frac{1}{2}a)^{2m} {}_2F_1(-m, -\mu-m, \nu+1; b^2/a^2)}{\lfloor m \; \Gamma(\mu+m+1)(\lambda+\mu+\nu+2m+1)} \quad .. \tag{3·3}$$

and taking $\mu = \frac{1}{2}, \lambda = \frac{3}{2}$ we get

$$\int_0^1 X \sin aX\, J_\nu(bX)\, dX = \frac{(\frac{1}{2} a)^{\frac{1}{2}} (\frac{1}{2} b)^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m .$$

$$\cdot \frac{(\frac{1}{2} a)^{2m}\; {}_2F_1(-m, -\frac{1}{2} - m; \nu + 1; b^2/a^2)}{\lfloor m \quad \Gamma(m + \frac{3}{2})\; (\nu + 2m + 3)} \quad \ldots \quad \ldots \quad (3\cdot4)$$

and taking $\mu = -\frac{1}{2}, \lambda = \frac{3}{2}$, we get

$$\int_0^1 X \cos aX\, J_\nu(bX)\, dX = \frac{(\frac{1}{2} a)^{-\frac{1}{2}} (\frac{1}{2} b)^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m .$$

$$\cdot \frac{(\frac{1}{2} a)^{2m}\; {}_2F_1(-m, \frac{1}{2} - m; \nu + 1, b^2/a^2)}{\lfloor m \quad \Gamma(m + \frac{1}{2})\; (\nu + 2m + 2)} \quad \ldots \quad \ldots \quad (3\cdot5)$$

Let $\mu = o, a = \sqrt{c^2 + d^2 - 2cd \cos \phi}$ in (3·3) (3·6)

Integrating between 0 and π with respect to ϕ we get

$$\int_0^1 X\lambda J_\nu(bX) J_0(cX) J_0(dX) dX = 2 \frac{(\frac{1}{2}b)^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m \frac{1}{2^{2m} (\lfloor m)^2\, \Gamma(\nu+\lambda+2m+1)}\, \mathrm{I} \quad (3\cdot7)$$

where

$$\mathrm{I} = \frac{1}{\pi} \int_0^\pi (c^2+d^2-2cd \cos\phi)^m\; {}_2F_1\left(-m, -m; \nu+1, \frac{b^2}{c^2+d^2-2cd \cos\phi}\right) d\phi$$

$$= \frac{1}{\pi} \sum_{r=0}^{m} \frac{(-m)_r\, (-m)_r}{(\nu+1)_r\, r!}\, b^{2r} \int_0^\pi (c^2 + d^2 - 2cd \cos \phi)^{m-r}\, d\phi$$

$$= \sum_{r=0}^{m} \frac{(-m)_r\, (-m)_r}{(\nu+1)_r\, r!}\, b^{2r}\, (c^2 \sim d^2)^{m-r}\, P_{m-r}\left(\frac{c^2+d^2}{c^2 \sim d^2}\right) \quad \ldots \quad (3\cdot8)$$

where $P_n(z)$ are usual Legendre's Polynomials.

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