SOME INFINITE SERIES INVOLVING LEGENDRE FUNCTIONS

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The sums of some interesting infinite series involving Legendre functions have been derived. Some particular cases have also been obtained.

The object of this note is to derive the sums of some interesting infinite series involving Legendre functions. The results proved are as under:

(i) $$\sum P_{m+\nu}^{\mu}\left(\frac{\xi+1/\xi}{2}\right)\frac{(\delta)_m\,\xi^{-m}}{m!}$$

$$=\frac{\Gamma(\frac{1}{2}-\mu-\delta)}{\Gamma(1-2\mu-\delta)\sqrt{\pi}}\left(1-1/\xi^2\right)^{-(\delta+\mu)}\xi^{-(\nu+1)}$$

$$F\left(\tfrac{1}{2}-\mu,\ 1+\nu-\mu-\delta;\ 1-2\mu-\delta;\ 1-1/\xi^2\right),\quad R\left(1-\frac{1}{\xi^2}\right)<1 \qquad (1)$$

(ii) $$\sum P_{m+\mu+1}^{-\mu}\left(\frac{\xi+\frac{1}{\xi}}{2}\right)\frac{(\delta)_m\,(y\,\xi)^m}{m!}$$

$$=2^{-(2\mu+1)}\ \xi^{-(\mu+1)}\ (1-y)^{-\delta}\ (\xi^2-1)^{\mu}$$

$$\times\left\{F\left[\mu+\tfrac{1}{2},\ \delta;\ 2\mu+2;\ \frac{y(\xi^2-1)}{1-y}\right]+\xi^2 F\left[\mu+\frac{3}{2},\delta;\ 2\mu+2;\ \frac{y(\xi^2-1)}{1-y}\right]\right\}$$

$$R\left(\frac{y(\xi^2-1)}{1-y}\right)<1 \qquad (2)$$

(iii) $$\sum_{m=0}^{\infty}\frac{\Gamma(m+\mu+\nu+1)}{\Gamma(m+1)}(\xi t)^m\ P_{m+\nu}^{-\mu}\left(\frac{\xi+1/\xi}{2}\right)$$

$$=\frac{\Gamma(\mu+\nu+1)}{R^{\nu+1}}\ P_{\nu}^{-\mu}\left(\frac{1+\xi^2-2t\xi^2}{2\xi R}\right)$$

$$t\,\xi<\min\left(\xi,\frac{1}{\xi}\right) \qquad (3)$$

where $R=[(1-t)(1-t\xi^2)]^{\frac{1}{2}}$

(iv) $$\sum_{m=0}^{\infty}\frac{(\delta)_m}{(1+2\alpha)_m}\ C_m^{\alpha+\frac{1}{2}}(z)\ y^m$$

$$=(1-zy)^{-\delta}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}\delta,\ \frac{1}{2}\delta+\frac{1}{2};\\ 1+\alpha;\end{matrix}\ \frac{y^2(z^2-1)}{(1-yz)^2}\right],\quad Rl\left(\frac{y^2(z^2-1)}{(1-yz)^2}\right)<1 \qquad (4)$$

The equations (1) to (4) are derived from the more general result, viz.,

$$\Gamma(\mu+1)\, 2^{2\mu+1}\, \xi^{2\mu+\nu+1} (\xi^2-1)^{-\mu} \sum P^{-\mu}_{m+\nu}\left(\frac{\xi+1/\xi}{2}\right)\frac{(\delta)_m (y\,\xi)^m}{m!}$$

$$= F_2(\mu+\nu+1;\ \mu+\tfrac{1}{2},\ \delta;\ 2\mu+2,\ \mu+\nu+1;\ 1-1/\xi^2,\ y)$$
$$+ F_2(\mu+\nu+1;\ \mu+\tfrac{3}{2},\ \delta;\ 2\mu+2,\ \mu+\nu+1;\ 1-1/\xi^2,\ y) \qquad (5)$$

F_2 is the Appell's hypergeometric function.

It may be mentioned here that equation (3) is the same as the result derived by Truesdell[1], and equation (4) proved by Brafman. [cf. Erdelyi[2], p. 265]. Many more particular cases of equation (5) can be easily deduced but, for the sake of brevity, they are not given here.

Herein the Legendre function is defined by the relation,

$$\Gamma(\mu+1).\ 2^{2\mu+1}\, \xi^{2\mu+m+\nu+1} (\xi^2-1)^{-\mu}\, P^{-\mu}_{m+\nu}\left(\frac{\xi+1/\xi}{2}\right)$$
$$= 2F(\mu+\tfrac{1}{2},\ m+\nu+\mu+1;\ 2\mu+1;\ 1-1/\xi^2)$$
$$= F(\mu+\tfrac{1}{2},\ m+\nu+\mu+1;\ 2\mu+2;\ 1-1/\xi^2)$$
$$+ F(\mu+\tfrac{3}{2},\ m+\nu+\mu+1;\ 2\mu+2;\ 1-1/\xi^2) \qquad (6)$$

To prove equation (5) we note[3] that

$$\sum_m F(\alpha,\ \beta+m;\ \gamma;\ x).\ \frac{(\delta)_m y^m}{m!}$$
$$= \sum_{m,\,n} \frac{(\alpha)_n\ (\beta)_{m+n}\ (\delta)_m}{(\beta)_m\ (\gamma)_n}\ \frac{x^n\ y^m}{n!\,m!}$$
$$= F_2(\beta;\ \alpha,\ \delta;\ \gamma,\ \beta;\ x,\ y)$$
$$= (1-y)^{-\delta} F_1\left(\alpha;\ \beta-\delta,\ \delta;\ \gamma;\ x,\ \frac{x}{1-y}\right) \qquad (7)$$

Substituting Legendre functions for the hypergeometric functions with the help of equation (6), it is deduced that

$$\Gamma(\mu+1)\, 2^{2\mu+1}\, \xi^{2\mu+\nu+1} (\xi^2-1)^{-\mu} \sum P^{-\mu}_{m+\nu}\left(\frac{\xi+1/\xi}{2}\right)\frac{(\delta)_m (y\ \xi)^m}{m!}$$
$$= F_2(\mu+\nu+1;\ \mu+\tfrac{1}{2},\ \delta;\ 2\mu+2,\ \mu+\nu+1;\ 1-1/\xi^2,\ y)$$
$$+ F_2(\mu+\nu+1;\ \mu+\tfrac{3}{2},\ \delta;\ 2\mu+2,\ \mu+\nu+1;\ 1-1/\xi^2,\ y)$$
$$= (1-y)^{-\delta}\left[F_1\left(\mu+\tfrac{1}{2};\ \mu+\nu+1-\delta,\ \delta;\ 2\mu+2;\ 1-1/\xi^2,\ \frac{1-1/\xi^2}{1-y}\right)\right.$$
$$\left.+ F_1\left(\mu+\tfrac{3}{2};\ \mu+\nu+1-\delta,\ \delta;\ 2\mu+2;\ 1-1/\xi^2,\ \frac{1-1/\xi^2}{1-y}\right)\right] \qquad (8)$$

which is equation (5).

Particular Cases

Some particular cases are derived below:

(*i*) $y=1/\xi^2$

Since,

$$F_1'(\alpha, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\ \Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta')}\ F(\alpha, \beta; \gamma-\beta'; x) \qquad (9)$$

We can prove by elementary transformations[2,4] that

$$\sum P^{\mu}_{m+\nu}\left(\frac{\xi+{}^1/\xi}{2}\right)\frac{(\delta)_m\ \xi^{-m}}{m!}$$

$$= \frac{\Gamma(\frac{1}{2}-\mu-\delta)}{\Gamma(1-2\mu-\delta).\sqrt{\pi}}\ (1-{}^1/\xi^2)^{-(\delta+\mu)}\ \xi^{-(\nu+1)}$$

$$\times F(\tfrac{1}{2}-\mu, 1+\nu-\mu-\delta; 1-2\mu-\delta; 1-z/\xi^2) \qquad (10)$$

(*ii*) $\nu=\mu+1$;

Noting that

$$F_1(\alpha; \beta, \beta'; \beta+\beta', x, y) = (1-y)^{-\alpha}\ F\left(\alpha, \beta; \beta+\beta'; -\frac{x-y}{1-y}\right) \qquad (11)$$

It is easily seen[3] that

$$\sum P^{-s}_{m+\mu+1}\left(\frac{\xi+1/\xi}{2}\right)\cdot\frac{(\delta)_m\ (y\ \xi)^m}{m!}$$

$$= \frac{2^{-(2\mu+1)}\ \xi^{-(\mu+1)}\ (1-y)^{-\delta}\ (\xi^2-1)^{\mu}}{\Gamma(\mu+1)}$$

$$\times\left\{F\left[\mu+\tfrac{1}{2}, \delta; 2\mu+2; \frac{y\ (\xi^2-1)}{1-y}\right]+\xi^2\ F\left[\mu+\tfrac{3}{2}, \delta; 2\mu+2; \frac{y\ (\xi^2-1)}{1-y}\right]\right\} \qquad (12)$$

(*iii*) $\delta = \mu+\nu+1$:

From the result,

$$F_2(\alpha; \beta, \beta'; \beta, r'; x,y) = (1-x)^{-\alpha}\ F\left(\alpha, \beta'; r'; \frac{y}{1-x}\right). \qquad (13)$$

one obtains, after some rearrangements,

$$\sum_{m=o}^{\infty}\frac{\Gamma(m+\mu+\nu+1)}{\Gamma(m+1)}(\xi\ t)^m\ P^{-\mu}_{m+\nu}\left(\frac{\xi+1/\xi}{2}\right)$$

$$= \frac{\Gamma(\mu+\nu+1)}{R+1}\ P^{-\mu}_{\nu}\left(\frac{1+\xi^2-2\ t\ \xi^2}{2\ \xi\ R}\right) \qquad (14)$$

where

$$R = [(1-t)\ (1-t\ \xi^2)]^{\frac{1}{2}}$$

(*iv*) By simple manipulations, it can be proved that,

$$\sum P_{m+v}^{\mu}(z).\ \frac{(\delta)_m y^m}{m!}$$

$$= \frac{2^{\mu}\,(z^2-1)^{-1/2\mu}}{\Gamma(1-\mu)} \cdot (z+\sqrt{z^2-1})^{\mu-1-\nu}$$

$$\times F_2\left(1+\nu-\mu;\tfrac{1}{2}-\mu,\delta;1-2\mu,1+\nu-\mu;\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}},\frac{y}{z+\sqrt{z^2-1}}\right) \quad (15)$$

An interesting particular case is obtained when $\nu = -\mu$ observing that,

$$F\,(a, a+\tfrac{1}{2}, c; z) = (1+z^{\frac{1}{2}})^{-2a} F\left(2a, c-\tfrac{1}{2}, 2c-1; \frac{2z^{\frac{1}{2}}}{1+z^{\frac{1}{2}}}\right) \quad (16)$$

and using equation (11) one obtains,

$$\sum_{m=o}^{\infty} C_m^{\alpha+\frac{1}{2}}(z)\ \frac{(\delta)_m\ y^m}{(1+2\alpha)_m}$$

$$= (1-z\ y)^{-\delta}\ {}_2F_1\left[\begin{matrix}\tfrac{1}{2}\delta,\ \tfrac{1}{2}\delta+\tfrac{1}{2};\\ 1+\alpha;\end{matrix}\ \frac{y^2(z^2-1)}{(1-yz)^2}\right] \quad (17)$$

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