EFFECT OF SMALL VARIATIONS IN ROCKET DESIGN PARAMETERS ON ITS PERFORMANCE

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The effects of small changes in the design variables of a rocket on its flight performance—all-burnt velocity and height at all-burnt are deduced. Also the effects on all-burnt height when all-burnt velocity is kept constant by changing either (*i*) payload weight or (*ii*) propellant weight or (*iii*) thrust separately have been found with changes in other design parameters.

The effects of small perturbations of rocket parameters, such as specific impulse, payload weight, propellant weight, structure weight and thrust, on all-burnt velocity and burn-out altitude have been obtained analytically for a single stage rocket. For a rocket already in existence or to be designed, a possible combination of design parameters, which may give optimum all-burnt velocity, has sometimes to be found. Such combinations have been obtained by finding the change required in a particular parameter when another parameter undergoes a small change and the velocity at all-burnt is kept unaltered. The velocity at all-burnt can be made constant by changing either (*i*) payload weight or (*ii*) propellant weight or (*iii*) thrust. The effects on altitude at all-burnt due to changes in parameters have been calculated in all the three cases and the three sets of results have been compared numerically.

The primary factors that contribute to the velocity and height at all-burnt are: payload weight, propellant weight, structure weight, specific impulse, thrust and duration of burning. Analytical expressions for velocity and altitude at all-burnt in the case of negligible earth's atmosphere* are given by

$$V_b = gI_{sp} \log r - gt_b \tag{1}$$

$$h_b = gI_{sp}\, t_b \left(1 + \frac{\log r}{1-r}\right) - \tfrac{1}{2} g t_b^2 \tag{2}$$

where

$$r = \frac{M_p + M_s + M_L}{M_s + M_L} \tag{3}$$

M_p = propellant weight, M_s = structure weight, M_L = payload weight, t_b = total time of burning and I_{sp} = specific impulse.

CHANGES IN PARAMETERS

The effects of any possible changes in design variables on the flight performance can be discussed qualitatively from first principles as well as quantitatively from expressions (1) and (2).

*The true velocity and height obtained by taking atmosphere into account would differ from the one given here, but it is expected that the effect of changes of rocket parameters on these quantities would be much the same in both cases.

The time of burning in terms of rocket parameters is given by

$$t_b = \frac{M_p}{F} g I_{sp} \tag{4}$$

where F is the thrust obtained in the rocket.

Thus from (1), (2) and (4) it is obvious that V_b and h_b are direct functions of specific impulse I_{sp} and thus an increase (or decrease) in I_{sp} will result in an increase (or decrease) in V_b and h_b and these changes are given by

$$\frac{\partial V_b}{\partial I_{sp}} = g \left(\log r - \frac{W}{F} \cdot \frac{r-1}{r} \right) \tag{5}$$

$$\frac{\partial h_b}{\partial I_{sp}} = \frac{W}{F} \cdot \frac{r-1}{r} I_{sp} \left[2 \left(1 + \frac{\log r}{1-r} \right) - \frac{W}{F} \cdot \frac{r-1}{r} \right] \tag{6}$$

W being the total weight of the rocket.

Therefore for a unit per cent change in I_{sp} the change in V_b is unit per cent while the change in h_b is two per cent.

For a single stage rocket, changes in V_b and h_b due to equal changes in structure weight and payload weight are equivalent. A change in payload weight or structure weight does not have any effect on the time of burning and the effects are due only to the change in r and are given by

$$\frac{\partial V_b}{\partial M_L} = \frac{\partial V_b}{\partial M_s} = - g I_{sp} \frac{r-1}{M_o}, \tag{7}$$

where M_o is the initial mass of the rocket.

$$\frac{\partial h_b}{\partial M_L} = \frac{\partial h_b}{\partial M_s} = \frac{g I^2_{sp}}{F} \frac{r-1}{r} \left(1 + \frac{r \log r}{1-r} \right) \tag{8}$$

Since $r > 1$, an increase (or decrease) in payload weight or structure weight leads to a decrease (or increase) in velocity and height at all-burnt. Table 1 and Fig. 1 give the percentage change in h_b due to unit per cent change in M_L or M_s for different values of r and F/W.

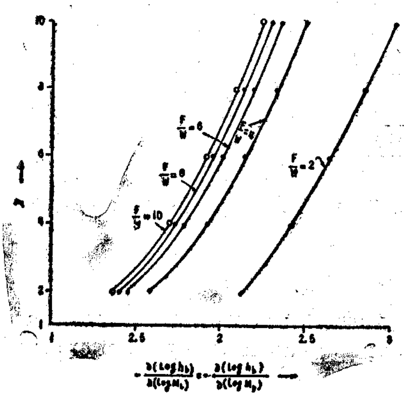

FIG. 1—Percentage changes in h_b due to unit per cent change in M_L or M_s for different values of r and F/W.

Again any possible increase (or decrease) in the propellant weight without other changes will result in increase (or decrease) in the time of burning as well as in the value of r and the expression for change in V_b is given by

$$\frac{\partial V_b}{\partial M_p} = \frac{g I_{sp}}{M_o} \left(1 - \frac{W}{F} \right) \tag{9}$$

which is a positive quantity.

Also, since the burning time is increased (or decreased) with an increase (or decrease) in M_p, the thrust will be acting for a longer (or shorter) duration and h_b will increase (or decrease) according as

$$\frac{\partial h_b}{\partial M_p} = \frac{g^2 I^2_{sp}}{F} \frac{r-1}{r} \left(1 - \frac{W}{F} \right) \tag{10}$$

which otherwise is a positive quantity.

TABLE 1

PERCENTAGE CHANGE IN h_b FOR CHANGES IN M_L OR M_s

F/W \ r	2	4	6	8	10
2	—2·123	—2·420	—2·651	—2·842	—3·016
4	—1·580	—1·909	—2·137	—2·319	—2·478
6	—1·456	—1·784	—2·008	—2·184	—2·339
8	—1·401	—1·727	—1·948	—2·123	—2·276
10	—1·370	—1·695	—1·915	—2·088	—2·239

TABLE 2

PERCENTAGE CHANGE IN h_b FOR CHANGE IN M_p

F/W \ p	2	4	6	8	10
2	1·374	1·070	0·961	0·904	0·868
4	1·534	1·266	1·163	1·106	1·069
6	1·571	1·315	1·213	1·157	1·122
8	1·587	1·337	1·237	1·181	1·146
10	1·596	1·349	1·250	1·195	1·160

Table 2 and Fig. 2 give the values for percentage changes in h_b for unit percent change in M_p for various values of r and F/W.

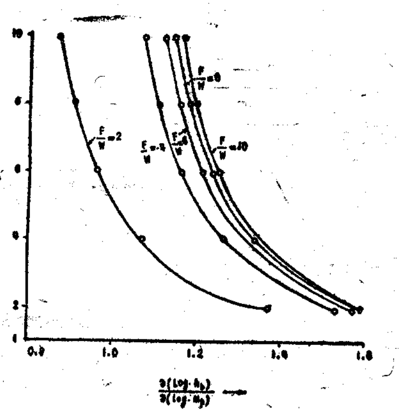

FIG. 2—Percentage changes in hp for unit per cent change in Mp for various values of r and F/W.

Finally, a change in thrust without any other alterations in the physical parameters of the rocket will only affect the time of burning of the propellant. Consequently an increase (or decrease) in F will decrease (or increase) the time of burning and thereby increase (or decrease) the all-burnt velocity but decrease (or increase) the all-burnt height. The two changes are given by

$$\frac{\partial V_b}{\partial F} = g \cdot \frac{I_{sp}}{F} \cdot \frac{W}{F} \cdot \frac{r-1}{r} \quad (11)$$

$$\frac{\partial h_b}{\partial F} = -g \cdot \frac{I^2_{sp}}{F} \cdot W \cdot \frac{r-1}{r} \cdot \left[\left(1 + \frac{\log r}{1-r}\right) - \frac{W}{F} \cdot \frac{r-1}{r}\right] \quad (12)$$

Table 3 and Fig 3 give the percentage change in h_b for unit percent change in F for given values of r and F/W.

TABLE 3

PERCENTAGE CHANGE IN h_b FOR CHANGE IN F

F/W \ r	2	4	6	8	10
2	—0·313	—0·465	—0·519	—0·548	—0·562
4	—0·744	—0·789	—0·806	—0·816	—0·802
6	—0·843	—0·869	—0·879	—0·884	—0·896
8	—0·887	—0·905	—0·912	—0·916	—0·918
10	—0·911	—0·925	—0·931	—0·934	—0·936

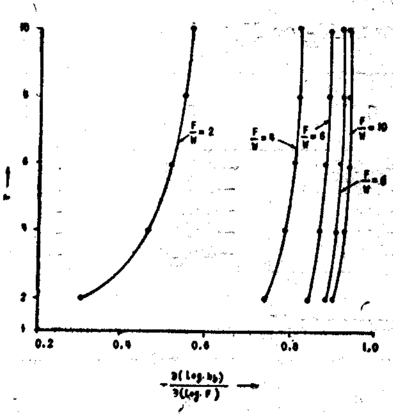

FIG. 3—Percentage change in h_b for unit percent change in F for different values of r and F/W.

Changes in all-burnt height when the velocity at all-burnt is kept constant—

To find an optimum combination of parameters it is important from the point of view of the designer to discuss the benefit of a change in one parameter over a change in another. Seifert and Summerfield have discussed one particular case—that of finding for zero gravitational field the change in I_{sp} for a given change in structure factor so that no change in burnt-out velocity occurs. The velocity can be brought to its given value by making necessary changes in (*i*) payload weight or (*ii*) propellant weight or (*iii*) thrust. But in every case the height reached at all-burnt will be affected. The required changes in parameters can be obtained and the effect on height reached in each case can be calculated. The changes can be represented by subscripts $(\quad)_{V_b,\,M_L}$ / $(\quad)_{V_b,\,M_P}$ and $(\quad)_{V_b,\,F}$ meaning thereby that velocity V_b has been kept fixed by altering M_L M_P and F respectively.

(*a*) *Velocity is kept constant by varying payload weight*—The required change in payload or structure weight due to a change in I_{sp}, in order that V_b maintains a given value, is given by

$$\left(\frac{M_L}{\partial I_{sp}}\right)_{V_b} = \frac{M_o\left\{\log r - \frac{W}{F}\cdot\frac{r-1}{r}\right\}}{(r-1)\,I_{sp}} \tag{13}$$

which is a positive quantity and M_L has to be increased with increase in I_{sp}. The effect of these two changes on h_b is given by

$$\left(\frac{\partial h_b}{\partial I_{sp}}\right)_{V_b,\,M_L} = \frac{W}{F}\cdot\frac{r-1}{r}\,g I_{sp}\left[\left(1+\frac{\log r}{1-r}\right)\left(2-\frac{W}{F}\right)-\frac{\log r}{1-r}\left(1+\frac{r\log r}{1-r}\right)\right] \tag{14}$$

Again an increase (or decrease) in structure weight M_s will mean a corresponding decrease (or increase) in M_L for V_b to remain invariable and thus the net effect on the height reached will be nil. Thus

$$\left(\frac{\partial M_L}{\partial M_s}\right)_{V_b} = -1 \tag{15}$$

and

$$\left(\frac{\partial h_b}{\partial M_s}\right)_{V_b,\,M_L} = o \tag{16}$$

Further, since for an increase (or decrease) in M_p there is an increase (or decrease) in V_b, it is clear from (7) that M_L must be increased by an amount given by

$$\left(\frac{\partial M_L}{M_p}\right)_{V_b} = \frac{1-\frac{W}{F}}{r-1} \tag{17}$$

and the net effect on all-burnt height will be

$$\left(\frac{\partial h_b}{\partial M_p}\right)_{V_b,\,M_L} = \frac{g I^2_{sp}}{M_o}\cdot\frac{W}{F}\cdot\frac{r-1}{r}\left[1+\frac{1}{r-1}\left(1+\frac{r\log r}{1-r}\right)\right] \tag{18}$$

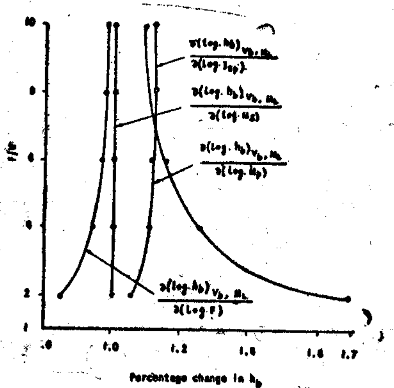

FIG. 4—Percentage change in h_b for different values of F/W due to unit percent changes in I_{sp}, M_p, M_s and F keeping V_b constant by varying M_L.

We have seen in (11) that for an increase (or decrease) in F, V_b is increased (or decreased) and consequently it follows from (7) that to bring it back to its original value we must increase (or decrease) M_L; the requirement on M_L is given by

$$\left(\frac{\partial M_L}{\partial F}\right)_{V_b} = \frac{1}{rg}\left(\frac{W}{F}\right)^2 \qquad (19)$$

and the overall effect on all-burnt height is

$$\left(\frac{\partial h_b}{\partial F}\right)_{V_b,\, M_L} = -\frac{M_p}{F^2}\, g^2 I^2_{sp}\left(1+\frac{\log r}{1-r}\right)\left(1-\frac{W}{F}\right) \qquad (20)$$

Table 4 and Fig. 4 give the percentage change in h_b for a given value of $r = 2$ and for different values of F/W due to unit percent changes in the various parameters.

(*b*) *Velocity is kept constant by changing propellant weight*—In this case we see from (5) and (9) that in order that all-burnt velocity may remain fixed due to a change in I_{sp}, M_p should be changed according to

$$\left(\frac{\partial M_p}{\partial I_{sp}}\right)_{V_b} = -\frac{M_o\left\{\log r - \frac{W}{F}\cdot\frac{r-1}{r}\right\}}{I_{sp}\left(1-\frac{W}{F}\right)} \qquad (21)$$

which is a negative quantity, and M_p should decrease for a given increase in I_{sp} and *vice versa*, the total change in height reached will be

$$\left(\frac{\partial h_b}{\partial I_{sp}}\right)_{V_b,\, M_p} = g\, I_{sp}\,\frac{W}{F}\cdot\frac{r-1}{r}\left[\,2\left(1+\frac{\log r}{1-r}\right) - \log r\,\right] \qquad (22)$$

TABLE 4

PERCENTAGE CHANGE IN h_b FOR $r = 2$

Parameter \ F/W	2	4	6	8	10
I_{sp}	1·059	1·102	1·112	1·116	1·118
M_s	—1·000	—1·000	—1·000	—1·000	—1·000
M_p	1·687	1·256	1·157	1·113	1·089
F	0·844	0·942	0·964	0·974	0·979

Comparing (14) and (22), we observe that the change in height in the first case is more than that in the second by an amount given by

$$\frac{W}{F} g I_{sp} \left(1 + \frac{\log r}{1-r}\right) \left(\log r - \frac{r-1}{r} \cdot \frac{W}{F}\right)$$

Next, when there is a change in M_L or M_s, then in order that V_b remains undisturbed, the requisite change in M_p is

$$\left(\frac{\partial M_p}{\partial M_L}\right)_{V_b} = \left(\frac{\partial M_p}{\partial M_s}\right)_{V_b} = \frac{r-1}{1-\frac{W}{F}} \tag{23}$$

and is a positive quantity.

The change in the all-burnt height will be

$$\left(\frac{\partial h_b}{\partial M_L}\right)_{V_b, M_p} = \left(\frac{\partial h_b}{\partial M_s}\right)_{V_b, M_p} = \frac{g^2 I^2_{sp}}{F} \left(1 + \frac{\log r}{1-r}\right) (r-1) \tag{24}$$

If we compare (15) and (23), we notice that in the first case M_L has to be decreased with increase in M_s for velocity to be uneffected, while in the second case, M_p has to be increased. But a comparison of (16) and (24) shows that this increase of M_p results in giving more vertical height.

Again due to a change in F, in order that V_b may remain constant, the required change in M_p is obtained from

$$\left(\frac{\partial M_p}{\partial F}\right)_{V_b} = -\frac{1}{g}\left(\frac{W}{F}\right)^2 \cdot \frac{r-1}{r} \cdot \frac{1}{1-\frac{W}{F}} \tag{25}$$

and thus M_p must be decreased for an increase in F. But these two changes are reflected in h_b according as

$$\left(\frac{\partial h_b}{\partial F}\right)_{V_b, M_p} = \frac{W}{F} \cdot \frac{g I^2_{sp}}{F} \left(1 + \frac{\log r}{1-r}\right) \tag{26}$$

Table 5 and Fig. 5 give percentage changes in all-burnt height due to unit per cent change in various parameters for $r = 2$ and different values of F/W. σ represents the ratio M_L/M_p.

(c) *Velocity kept constant by controlling the thrust*—Due to a change in I_{sp}, the requirement on F for a constant all-burnt velocity is given by

$$\left(\frac{\partial F}{\partial I_{sp}}\right)_{V_b} = -\frac{F\left\{\log r - \frac{W}{F} \cdot \frac{r-1}{r}\right\}}{\frac{W}{F} \cdot \frac{r-1}{r}} \tag{27}$$

TABLE 5

PERCENTAGE CHANGES IN ALL-BURNT HEIGHT

Parameter \ F/W	2	4	6	8	10
I_{sp}	—0·436	—0·324	—0·299	—0·287	—0·281
M_L or M_s	1·687σ	1·256σ	1·157σ	1·113σ	1·089σ
F	3·374	2·511	2·314	2·227	2·177

and thus thrust is required to be lowered for an increase in I_{sp}. The overall effect on h_b is

$$\left(\frac{\partial h_b}{\partial I_{sp}}\right)_{V_b,\,F} = g\, I_{sp}\left[\frac{W}{F}\cdot\frac{r-1}{r} + \log r\left(1-\frac{W}{F}\right) + \frac{\log^2 r}{1-r}\right] \quad (28)$$

Again the change required in F so that V_b is unchanged due to variation of M_p is

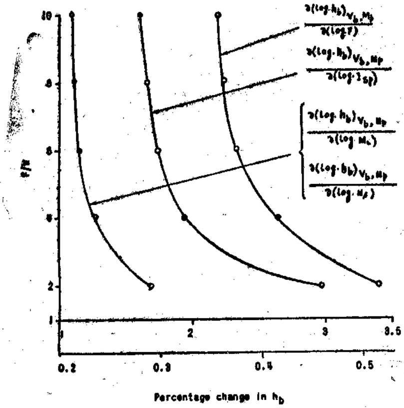

FIG. 5—Percentage changes in h_b due to unit percent changes in I_{sp}, M_L or M_s and F for different values of F/W keeping V_b constant by varying M_p.

$$\left(\frac{\partial F}{\partial M_p}\right)_{V_b} = -\frac{F^2\left(1-\dfrac{W}{F}\right)}{M_o\, M_p\, g} \quad (29)$$

which is the reciprocal of the change in M_p required due to change in F when V_b is kept invariable by changing M_p. In this case the effect on h_b is

$$\left(\frac{\partial h_b}{\partial M_p}\right)_{V_b,\,F} = \frac{g^2 I^2_{sp}}{W}\left(1-\frac{W}{F}\right)\left(1+\frac{\log r}{1-r}\right) \quad (30)$$

Finally, as mentioned before, for a single stage rocket the change in M_L and M_s have equivalent effect when other parameters are unaffected and thus the requirement on F due to change in M_L or M_s is

$$\left(\frac{\partial F}{\partial M_L}\right)_{V_b} = \left(\frac{\partial F}{\partial M_s}\right)_{V_b} = \frac{F^2}{W\,(M_s + M_L)} = rg\left(\frac{F}{W}\right)^2 \quad (31)$$

which is a positive quantity.

The change in all-burnt height in this case is

$$\left(\frac{\partial h_b}{\partial M_L}\right)_{V_b,F} = \left(\frac{\partial h_b}{\partial M_s}\right)_{V_b,F} = -\frac{(r-1)\, g^2\, I^2_{sp}}{W}\left(1+\frac{\log r}{1-r}\right)\left(1-\frac{W}{F}\right) \quad (32)$$

Table 6 and Fig. 6 give the percentage changes in h_b for unit per cent changes in parameters for $r=2$ and different values of F/W.

TABLE 6

PERCENTAGE CHANGES IN H_b FOR UNIT PERCENT CHANGE IN PARAMETER FOR $r = 2$

Parameter \ F/W	2	4	6	8	10
I_{sp}	2·554	5·383	8·619	10·946	13·721
M_p	1·687	1·256	1·157	1·113	1·089
M_L or M_s	$-1{\cdot}687\sigma$	$-1{\cdot}256\sigma$	$-1{\cdot}157\sigma$	$-1{\cdot}113\sigma$	$-1{\cdot}089\sigma$

TABLE 7

PERCENTAGE CHANGES IN h_b FOR UNIT PERCENT CHANGE IN V_b

I_{sp}	M_L or M_s	M_p	F
2·000	0·941	2·436	—0·554

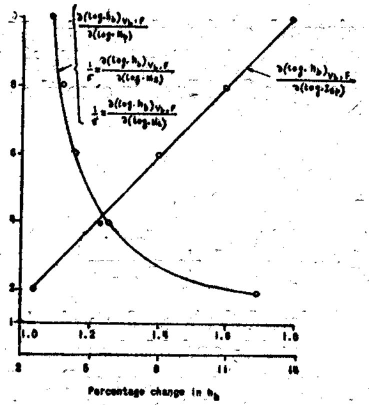

FIG. 6—Percentage changes in h_b for unit per cent changes in I_{sp}, M_p, M_L or M_s for different values of F/W keeping V_b constant by varying F.

Effect on all-burnt height due to variation in all-burnt velocity

In this case the subscripts $(\ \ )I_{sp}$, $(\ \ )M_L$, $(\ \ )M_s$, $(\ \ )M$ and $(\ \ )F$ represent the various changes due to small errors in V_b which can be caused by small errors in I_{sp}, M_L, M_s, M_p and F respectively. The changes in all-burnt altitude are given by the following expressions:

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{I_{sp}} = \frac{r-1}{r} \cdot \frac{W}{F} I_{sp} \left[2\left(1 + \frac{\log r}{1-r}\right) - \frac{W}{F}\frac{r-1}{r}\right] \left[\log r - \frac{W}{F}\frac{r-1}{r}\right]^{-1} \quad (33)$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{M_L} = \left(\frac{\partial h_b}{\partial V_b}\right)_{M_s} = -\frac{1}{r}\frac{W}{F} I_{sp}\left(1 + \frac{r \log r}{1-r}\right) \quad (34)$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{M_p} = \frac{r-1}{r} \cdot \frac{W}{F} I_{sp} \quad (35)$$

$$\left(\frac{\partial h_b}{\partial V_b}\right)_{F} = -I_{sp}\left[\left(1 + \frac{\log r}{1-r}\right) - \frac{r-1}{r} \cdot \frac{W}{F}\right] \quad (36)$$

Table 7 gives the values of percentage changes in h_b due to unit per cent changes in V_b brought about by changes in I_{sp}, M_L or M_s, M_p and F respectively, for $r=2$ and $F/W = 2$.

h_b increases due to an increase in V_b obtained by changing I_{sp}, M_L or M_s and M_p but decreases when V_b changes due to an increase in F. An increase in F which brings an increase in V_b results in a decrease of all-burnt altitude.

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