TWO-BIN INVENTORY MODEL WITH GENERAL LEAD TIME

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An inventory model is considered in which two integers (Q, D) with $Q>D$ are defined such that whenever the stock level reaches D, immediately an order for Q items is made. Delivery takes place after a lapse of time. The sales are assumed to follow a Poisson distribution with mean rate λ.

Inventory as defined by Arrow[1], is a stock of goods which is held or stored for the purpose of future sale or production. Let us suppose that initially the level of the stock is M. As demand arises the stock level decreases. If the demand is constant throughout, we can know the exact time when the level will reach zero. But if the demand is stochastic, the time at which the stock level will come down to zero cannot be determined exactly. Thus if the demand or lead time or both vary according to a certain distribution, it may happen that sooner or later the stock comes down to zero and the customer will have to leave dissatisfied. This situation is highly unacceptable in Defence and business. On the other hand, it is not possible to keep large stocks available because of the storage charges and the capital tied down. The problem is then to devise an optimal policy of maintaining and keeping the cost of the inventory to a minimum.

In this paper, we consider a simple inventory policy in which the inventory M is divided into two parts, the buffer stock, D and the ordinary usage stock, Q so that $M=D+Q$ where $Q>D$. The buffer stock can be considered as a 'second-bin' which will not be utilized until the entire ordinary usage stock has been exhausted and then an order for Q items is made and the units of the second-bin are available for sale. The purpose of the buffer stock is, of course, to satisfy customers demand during the 'lead time', *i.e.*, the time that elapses between the placing of the order of the store and the receipt of the same. This policy it, therefore, known as the two-bin inventory policy. It may be mentioned that this problem has been discussed by Karlin[1] and Gaver[2] using the renewal theory approach.

THE PROBLEM

We assume that demands follow a Poisson process with mean rate λ, so that the probability that exactly n demands occur in time t is given by

$$a_n(t) = \frac{(\lambda t)^n}{n!} \cdot e^{-\lambda t}$$

where λ is the demand rate. It is also assumed that if no stock is on hand no demand will wait, *i.e.*, no back ordering will occur. This is known as a lost sales case. The lead time is assumed to follow a general distribution with probability density $S(x)$.

DEFINITIONS

Let $P_n(t)$ be the probability that there are n units in the system at time t, where $D < n \leqslant M$

$Q_n(x,t)\,dx$ the probability that at time t, there are n units in the system $(0 \leqslant n \leqslant D)$ and the elapsed lead time lies between x and $x+dx$.

$\eta(x)\,dx$ the first order probability that the order arrives between time x and $x+dx$, conditioned that it has not arrived up to x and is related to the probability density $S(x)$ by means of the relation:

$$S(x) = \eta(x)\, exp\left\{ -\int_o^{\infty} \eta(x)\, dx \right\}$$

The equations governing the process are:

$$P_n(t+\triangle) = P_n(t)\{1-\lambda\triangle\} + \lambda\triangle P_{n+1}(t) + \int_o^{\infty} Q_{n-Q}(x,t)\,\eta(x)\,dx, \qquad Q<n<M \tag{1}$$

$$P_M(t+\triangle) = P_M(t)\{1-\lambda\triangle\} + \int_o^{\infty} Q_{M-Q}(x,t)\,\eta(x)\,dx \tag{2}$$

$$P_n(t+\triangle) = P_n(t)\{1-\lambda\triangle\} + \lambda\triangle P_{n+1}(t) \qquad D<n<Q \tag{3}$$

$$Q_D(x+\triangle, t+\triangle) = Q_D(x,t)\{1-(\lambda+\eta(x))\triangle\} \tag{4}$$

$$Q_n(x+\triangle, t+\triangle) = Q_n(x,t)\{1-(\lambda+\eta(x))\triangle\} + \lambda\triangle Q_{n+1}(x,t) \qquad 1 \leqslant n < D \tag{5}$$

$$Q_o(x+\triangle, t+\triangle) = Q_o(x,t)\{1-\lambda\triangle\} + \lambda\triangle Q_1(x,t) \tag{6}$$

together with the boundary conditions

$$Q_D(o,t) = \lambda P_{D+1}(t) \tag{7}$$

$$Q_n(o,t) = o, \qquad o \leqslant n < D \tag{8}$$

SOLUTION

Assuming a steady state to exist, the solution of equations (4) and (5) with appropriate boundary conditions can be written as

$$Q_n(x) = Q_D(o)\,\frac{(\lambda x)^{D-n}}{(D-n)!}\, exp\left\{ -\left(\lambda x + \int_o^x \eta(x)\,dx\right) \right\}, \qquad 1 \leqslant n \leqslant D \tag{9}$$

because for $n = D$, the solution of (4) is

$$Q_D(x) = Q_D(o)\, exp\left\{ -\left(\lambda x + \int_o^x \eta(x)\,dx\right) \right\} \tag{10}$$

and substituting (10) in (5) for $n = D-1$, we get

$$Q_{D-1}(x) = Q_D(o)\,(\lambda x)\, exp\left\{ -\left(\lambda x + \int_o^x \eta(x)\,dx\right) \right\}$$

and similarly for $n = D-2, D-3, \ldots\ldots\ldots\ldots, 1$.

Using the value of $Q_1(x)$ from (9), we get from (6)

$$Q_\circ(x) = Q_D(o)\left[1 - exp(-\lambda x)\sum_{m=o}^{D-1}\frac{(\lambda x)^m}{m!}\right] exp\left\{-\int_o^x \eta(x)\,dx\right\} \tag{11}$$

Using equation (7), equation (3) gives

$$P_Q = P_{Q-1} = \ldots\ldots\ldots\ldots\ldots = P_{D+1} = \frac{1}{\lambda}\ \theta_D(o) \tag{12}$$

From equation (2) we get

$$\lambda P_M = \int_o^\infty Q_{M-Q}(x)\,\eta(x)\,dx$$

$$= \int_o^\infty Q_D(x)\,\eta(x)\,dx$$

$$= Q_D(o)\int_o^\infty \eta(x)\,e^{-(\lambda x + \int_o^x \eta(x)\,dx)}\,dx$$

$$= Q_D(o)\,A_0 \tag{13}$$

where Ao is defined as under:

Equation (1) for $n = M - 1$, gives

$$\lambda P_{M-1} = \lambda P_M + \int_o^\infty Q_{D-1}(x)\,\eta(x)\,dx$$

$$= Q_D(o)\,A_\circ + Q_D(o)\int_o^\infty e^{-\lambda x}\,\frac{\lambda x}{1!}\,S(x)\,dx$$

$$= Q_D(o)\,[A_\circ + A_1] \tag{14}$$

$$\text{where } A_i = \int_0^\infty e^{-\lambda x}\,\frac{(\lambda x)^i}{i!}\,S(x)\,dx \qquad (i = o, 1, 2, \ldots\ldots n-1) \tag{15}$$

Similarly we get

$$\lambda P_{M-2} = Q_D(o)\sum_{i=0}^{2} A_i \tag{16}$$

and in general

$$P_n = \frac{1}{\lambda}\ Q_D(o)\sum_{i=0}^{M-n} A_i \qquad Q < n < M \tag{17}$$

We have calculated all the probabilities in terms of $Q_D(o)$ which can be evaluated using the condition that the sum of all the probabilities is one

$$\int_0^\infty Q_0(x)\,dx + \int_0^\infty \sum_{n=1}^{D} Q_n(x)\,dx + \sum_{n=D+1}^{M} P_n = 1 \tag{18}$$

Hence

$$Q_D(o) = \lambda \div \left\{ \lambda T + Q - D + \sum_{i=0}^{D-1} (D-i) A_i \right\} \tag{19}$$

It can be varified that the results obtained above reduce to those obtained for the exponential or regular lead time distribution.

The probability P_0 that the system is out of stock is

$$P_0 = \int_0^\infty Q_o(x)\, dx$$

$$= \frac{\lambda T - D + \sum_{i=0}^{D-1} (D-i) A_i}{Q + \lambda T - D + \sum_{i=0}^{D-1} (D-i) A_i} \tag{20}$$

The quantities defined here will be required in determining the optimum values of criteria imposed by the management.

R = number of orders placed in time T.

L_s = average demands in the lead time period.

I = average unit inventory.

For the model mentioned earlier, we have

$$R = \int_o^\infty Q_o(x)\, dx + \int_o^\infty \sum_{n=1}^{D} Qn(x)\, dx$$

$$= \lambda T \div \left\{ Q + \lambda T - D + \sum_{i=o}^{D-1} (D-i) Ai \right\} \tag{21}$$

$$L_s = \lambda T \left(1 - \int_o^\infty Q_o(x)\, dx \right)$$

$$= \lambda TQ \div \left\{ Q + \lambda T - D + \sum_{i=o}^{D-1} (D-i) A_i \right\} \tag{22}$$

and

$$I = \int_{o}^{\infty} \sum_{n=1}^{D} n\, Q_n (x)\, dx + \sum_{n=D+1}^{Q} n\, P_n + \sum_{n=Q+1}^{M} n\, P_n$$

$$= \frac{\frac{1}{2} Q (Q+1) + Q \sum_{i=o}^{D-1} (D-i)\, A_i}{Q + \lambda T - D + \sum_{i=o}^{D-1} (D-i)\, A_i} \qquad (23)$$

DISCUSSION

Various criteria can be imposed to determine the optimum value of Q and D. We may decide to fix the value of P_o, the probability that the stock is out and then maximise the profit. Mathematically the problem is to determine Q and D in such a way that

$$P_o = \frac{\lambda T - D + \sum_{i=o}^{D-1} (D-i)\, A_i}{Q + \lambda T - D + \sum_{i=o}^{D-1} (D-i)\, A_i} \simeq \epsilon \qquad (24)$$

and the net gain per lead time is

$$G\, L_s - C_i\, I - Cp\, R = 1 \div \left\{ \lambda T + Q - D + \sum_{i=o}^{D-1} (D-i)\, A_i \right\} \times$$

$$\left\{ G\, \lambda T \cdot Q - Cp\, \lambda T - C_i\, \tfrac{1}{2} Q (Q+1) - C_i\, Q \sum_{i=o}^{D-1} (D-i)\, A_i \right\} \qquad (25)$$

where G is the gross profit per unit sold, C_i is the inventory cost per unit per year, C_p is the cost per replenishment order.

This is to be maximised. Since the parameters of the process, *i.e.*, L_s, I and R, involve complicated expression, maximisation poses a complicated problem. The solution may be obtained numerically. In the particular case, when the lead time is exponential, the result can be approximated as shown by Morse[3]. Various other criteria can be imposed and discussed as in Arrow[1] *et al*, but in all cases, the solutions will have to be found numerically.