SIMULATION OF THIRD ORDER SYSTEMS WITH SIMPLE-LEAD USING ONE OPERATIONAL AMPLIFIER

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A method has been developed for the simulation of third order linear systems in electronic analogue control and computation, with the aid of only one operational amplifier and a few two-terminal impedances. The circuits have been analysed and conditions of physical realisability discussed.

In the field of electronic analogue control and computation an engineer is often faced with the task requiring simulation of third order systems. The simulation of such systems is generally done with the aid of two or more operational amplifiers by well known techniques.

The author [1,2] has shown that it is possible to simulate third order linear systems with simple-lead using only one operational amplifier, three capacitors and five resistors. Of the six basic circuits that are possible, only three are presented here, the other three have already been discussed in detail elsewhere [3-5].

SIMULATION OF THIRD ORDER SYSTEMS

A network for the simulation of third order systems is shown in Fig. 1 and its transfer function has been shown[2] to be

$$\frac{E_o}{E_1} = -\frac{Y_1 Y_3 Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad (1)$$

A third order linear system with a simple-lead is characterised by a transfer function of the form

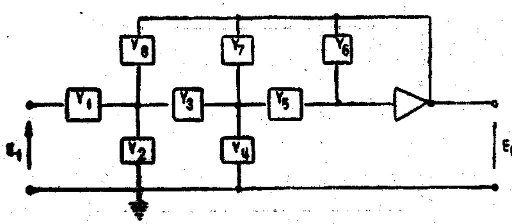

Fig. 1—Network for simulating third order systems.

$$F(s) = -\frac{b_1 s}{a_3 s^3 + a_2 s^2 + a_1 s + 1} \tag{2}$$

where b_1, a_1, a_2, a_3 are positive and real constants.

Simulation of the system as characterised by (2) with the network of Fig. 1, as seen from (1), would require at least three capacitors. Six basic circuits each employing three capacitors and five resistors are possible and only three of these will be discussed here.

(*a*) Y_1, Y_6 and Y_7 Capacitative

A network for simulating the system of (2) with

$$\left.\begin{aligned} Y_1 &= SC_1 \\ Y_6 &= SC_6 \\ Y_7 &= SC_7 \\ Y_2 &= Y_3 = Y_4 = Y_5 = \frac{1}{R} \\ Y_8 &= \frac{1}{aR} \end{aligned}\right\} \tag{3}$$

is shown in Fig. 2.

Substituting (3) in (1) and after simplifying, we get

$$\frac{E_o}{E_1} = -\frac{aRC_1S}{aR^3C_1C_6C_7S^3 + [3aR^2C_1C_6 + aR^2C_1C_7 + (2a+1)R_2C_6C_7]\,S^2 + [(5a+3)\,RC_6 + (2a+1)RC_7]S + 1} \tag{4}$$

Equations (2) and (4) will be identical if

$$b_1 = a\,T_1 \tag{5}$$

$$a_1 = (5\,a + 3)\,T_6 + (2a + 1)\,T_7 \tag{6}$$

$$a_2 = 3\,a\,T_1T_6 + a\,T_1T_7 + (2a + 1)\,T_6T_7 \tag{7}$$

$$a_3 = a\,T_1T_6T_7 \tag{8}$$

where

$$T_n = RC_n. \tag{9}$$

Simulation of the system of (2) with the network of Fig. 2 is possible only if the values of a, T_1, T_6, T_7 obtained as the solution of (5) through (8) are real and positive. It is

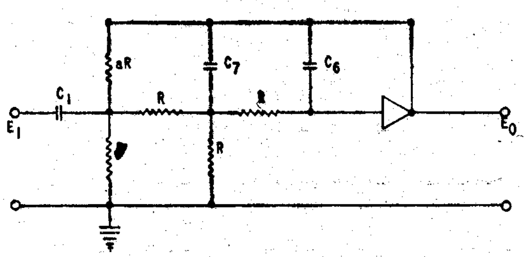

Fig. 2—Network for the simulation of $\dfrac{E_o}{E_1} = -\dfrac{b_1S}{a_3S^3 + a_2S^2 + a_1S + 1}$

required to determine, therefore, in terms of the given positive real a's and b's the values of a, T_1, T_6, T_7 and find the conditions, if any, under which these can be real and positive.

Elimination of a, T_1 and T_6 from (5) through (8) gives a quartic

$$2b_1^2\, T_7^4 - 2a_2 b_1^2\, T_7^3 + a_3 b_1\,(2a_1+6b_1+5)\, T_7^2 - a_3\,(a_3+5a_2 b_1)\, T_7 + 15a_3^2 b_1 = 0 \quad (10)$$

which, as is obvious, can have no negative real roots. Therefore, if its discriminant $\triangle$ is negative then (10) will have two positive real roots, and it can be shown in a manner similar to that discussed elsewhere[6] that if,

$$\left.\begin{aligned} a_1^2 &> \frac{12a_3}{b_1}. \\ a_2 &> \frac{a_3}{b_1}. \\ (a_2 b_1 - a_3)^2 &> 12a_3 b_1^3 \end{aligned}\right\} \quad (11)$$

and, either

$$OB > OQ > OA > OP \quad (12)$$

or

$$OQ > OB > OP > OA \quad (13)$$

where

$$OA = \frac{a_1 b_1 - \sqrt{a_1^2 b_1^2 - 12a_3 b_1}}{2b_1}$$

$$OB = \frac{a_1 b_1 + \sqrt{a_1^2 b_1^2 - 12a_3 b_1}}{2b^2_1}$$

$$OP = \frac{(a_2 b_1 - a_3) - \sqrt{(a_2 b_1 - a_3)^2 - 12a_3 b_1^3}}{2b_1^2}$$

$$OQ = \frac{(a_2 b_1 - a_3) + \sqrt{(a_2 b_1 - a_3)^2 - 12a_3 b_1^3}}{2b_1^2}$$

then one set of positive real a, T_1, T_6 and T_7 exists. But if $\triangle$ is negative and (11) is satisfied then two sets of positive real a, T_1, T_6 and T_7 exist for either one of which the circuit of Fig. 2 is physically realisable, provided that

$$OB > OQ > OP > OA \quad (14)$$

But, if its $\triangle$ is positive then (10) will have either no real roots or four positive real roots. Now, if $\triangle$ is positive and (11) is satisfied, then three sets of positive real values exist, if

$$OB > OQ > OA > OP \tag{15}$$

and two sets of positive real values exist, if

$$OQ > OB > OA > OP \tag{16}$$

and one positive real set of values exist, if

$$OQ > OB > OP > OA \tag{17}$$

Combining and summarising the inequalities of expressions (11) through (17); that is, if

$$\left.\begin{array}{l} Min \left[\dfrac{a^2_1}{12}, a_2 \right] > \dfrac{a_3}{b_1} \\ (a_2b_1 - a_3)^2 > 12\, a_3b_1^3 \end{array}\right\}$$

and, either

$$\left.\begin{array}{ll} & OB > OQ > OA > OP \\ \text{or} & \\ & OQ > OB > OP > OA \end{array}\right\} \tag{18}$$

then irrespective of whether the discriminant of (10) is positive or negative, at least one positive real set of a, T_1 T_6 and T_7 exists. But, if

$$\text{Either} \quad \left.\begin{array}{c} \Delta < 0 \\ OB > OQ > OP > OA \end{array}\right\}$$

$$\text{Or} \quad \left.\begin{array}{c} \Delta > 0 \\ OQ > OB > OA > OP \end{array}\right\}$$

then two sets of positive real a, T_1, T_6 and T_7 exist for either one of which the circuit of Fig. 2 is possible.

To summarise, if the inequalities of expression (11) and any one of the three expressions (14), (16) or (18) are satisfied, then it is possible to simulate the system of (2) with the network of Fig. 2. The network component values can be determined by solving for T_7 the quartic of (10) by methods that are well known and discussed at length in text-books on higher algebra[7]. Substitution of the positive real value(s) of T_7 so obtained into (5) through (8) gives on solution, the corresponding positive real value(s) of a, T_1 and T_6. Choosing arbitrarily a convenient value for one of the capacitors, the remaining component values can be obtained with the aid of (9) and (3).

(*b*) Y_3, Y_6 and Y_8 Capacitative

Another possible circuit for the simulation of the system of (2) is shown in Fig. 3, in which

$$\left.\begin{array}{r} Y_3 = SC_3 \\ Y_6 = SC_6 \\ Y_8 = SC_8 \\ Y_4 = Y_5 = Y_7 = \dfrac{1}{R} \\ Y_1 = Y_2 = \dfrac{1}{aR} \end{array}\right\} \tag{19}$$

Substituting (19) into (1) and simplifying, we get

$$\frac{E_0}{E_1} = -\frac{\frac{1}{2}RC_3S}{\frac{a}{2}R^3C_3C_6C_8S^3+\left[\left(\frac{3a+2}{2}\right)R^2C_3C_6+\frac{a}{2}R^2C_3C_8+\frac{3a}{2}R^2C_6C_8\right]S^2+\left[\frac{a}{2}RC_3+3RC_6+\frac{a}{2}RC_8\right]S+1} \quad (20)$$

Equations (2) and (20) will be identical if

$$b_1 = \tfrac{1}{2}T_3 \quad (21)$$

$$a_1 = \frac{a}{2}T_3 + 3T_6 + \frac{a}{2}T_8 \quad (22)$$

$$a_2 = \left(\frac{3a+2}{2}\right)T_3T_6 + \frac{a}{2}T_3T_8 + \frac{3a}{2}T_6T_8 \quad (23)$$

$$a_3 = \frac{a}{2}T_3T_6T_8 \quad (24)$$

where

$$T_n = RC_n \quad (25)$$

Elimination of a, T_3, and T_8 from (21) through (24) gives a cubic

$$9T_6^3 - (3a_1 + 2b_1)T_6^2 + a_2T_6 - a_3 = 0 \quad (26)$$

which can have no negative real roots and will have either one or three real positive roots depending on whether its discriminant $\triangle$ is positive or negative.

If $\triangle$ is positive, and

$$\left.\begin{aligned} &Min\left[\frac{a_1^2}{6},\ \frac{2}{3}a_2\right] > \frac{a_3}{b_1}. \\ &(2a_2b_1 - 3a_3)^2 > 32a_3b_1^3 \end{aligned}\right\} \quad (27)$$

and $$OB > OQ > OA > OP \quad (28)$$

where $$OA = \frac{(2a_2b_1 - 3a_3) - \sqrt{(2a_2b_1 - 3a_3)^2 - 32a_3b_1^3}}{8b_1^2}$$

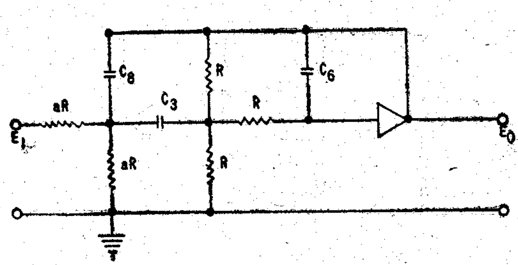

Fig. 3—Circuit with Y_3, Y_6 and Y_8 capacitative

$$OB = \frac{(2a_2b_1 - 3a_3) + \sqrt{(2a_2b_1 - 3a_3)^2 - 32a_3b_1^3}}{8b_1^2}$$

$$OP = \frac{a_1 b_1 - \sqrt{a_1^2 b_1^2 - 6a_3 b_1}}{6 b_1}$$

$$OQ = \frac{a_1 b_1 + \sqrt{a_1^2 b_1^2 - 6a_3 b_1}}{6b_1}.$$

then, as discussed elsewhere[6], one set of positive real a, T_3, T_6 and T_8 exists. But if $\triangle$ is negative; that is

$$\left.\begin{aligned} \Delta &= 4p^3 + 27q^2 < 0. \\ p &= \frac{a_2}{9} - \frac{1}{3}\left(\frac{3a_1 + 2b_1}{9}\right)^2 \\ q &= -\frac{a_3}{9} + \frac{a_2\,(3a_1 + 2b_1)}{243} - \frac{2}{27}\left(\frac{3a_1 + 2b_1}{9}\right)^3 \end{aligned}\right\} \qquad (29)$$

and (27) is satisfied, then one, two or three positive real sets can exist.

To summarise, if (27) and (28) are satisfied, then either one or three sets of positive real a, T_3, T_6 and T_8 exi t depending respectively on whether $\triangle$ is positive or negative. But if (27) and (29) are satisfied then at least one and possibly two or three sets of positive real values can exist.

(c) Y_5, Y_7 and Y_8 Capacitative

Another possible circuit is shown in Fig. 4, in which

$$\left.\begin{aligned} Y_5 &= SC_5 \\ Y_7 &= SC_7 \\ Y_8 &= SC_8 \\ Y_1 = Y_2 = Y_3 = Y_4 &= \frac{1}{R} \\ Y_6 &= \frac{1}{aR} \end{aligned}\right\} \qquad (30)$$

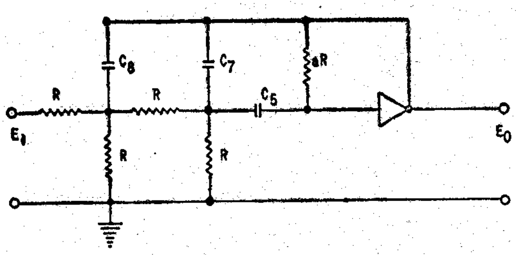

Fig. 4—Circuit with Y_5, Y_7 and Y_8 capacitative

Substituting (30) into (1) and simplifying, we get

$$\frac{E_0}{E_1} = -\frac{\frac{a}{5}\; RC_5S}{\frac{a}{5}\; R^3C_5C_7C_8S^3 + \frac{1}{5}\left[3aR^2C_5C_7 + (a+1)R^2C_5C_8 + R^2C_7C_8\right]S^2 + \frac{1}{5}\left[3RC_5 + 3RC_7 + 2RC_8\right]S + 1} \tag{31}$$

Equations (2) and (31) will be identical if

$$b_1 = \frac{a}{5}\, T_5 \tag{32}$$

$$a_1 = \frac{3}{5}\, T_5 + \frac{3}{5}\, T_7 + \frac{2}{5}\, T_8 \tag{33}$$

$$a_2 = \frac{3a}{5}\, T_5\, T_7 + \left(\frac{a+1}{5}\right) T_5\, T_8 + \frac{1}{5}\, T_7\, T_8 \tag{34}$$

$$a_3 = \frac{a}{5}\, T_5\, T_7\, T_8 \tag{35}$$

where

$$T_n = Rc_n \tag{36}$$

Elimination of a, T_5 and T_7 from (32) through (35) gives a cubic

$$2\, T_8^3 - 5\,(\, a_1 + 3\, b_1\,)\, T_8^2 + 15a_2\, T_8 - 45\, a_3 = o \tag{37}$$

which can have no negative real roots and will have either one or three positive real roots depending respectively on whether its discriminant is positive or negative. The situation in this case is very similar to that of case (b) already discussed, and it can be shown[6] that, if

$$\left.\begin{aligned} Min\left[\frac{25}{24}\, a_1^2,\, 5a_2\right] &> \frac{a_3}{b_1} \\ (\, 5a_2\, b_1 - a_3\,)^2 &> 300\, a_3\, b_1^3 \end{aligned}\right\} \tag{38}$$

and

$$O\,B > O\,Q > O\,A > O\,P \tag{39}$$

where

$$OA = \frac{(\, 5a_2b_1 - a_3\,) - \sqrt{(\, 5a_2b_1 - a_3\,)^2 - 300\, a_3b_1^3}}{10b_1^2}$$

$$OB = \frac{(\, 5a_2b_1 - a_3\,) + \sqrt{(\, 5a_2b_1 - a_3\,)^2 - 300\, a_3b_1^3}}{10\, b_1^2}$$

$$OP = \frac{5a_1b_1 - \sqrt{25a_1^2b_1^2 - 24a_3b_1}}{4b_1}$$

$$OQ = \frac{5a_1b_1 + \sqrt{25a_1^2b_1^2 - 24a_3b_1}}{4b_1}$$

then one set of positive real values exist if $\triangle$ is positive and three positive real sets of a, T_3, T_7 and T_8 exist if $\triangle$ is negative. But if (38) is satisfied and $\triangle$ is negative; that is

$$\Delta = 4p^3 + 27\,q^2 < o$$

where
$$\left.\begin{aligned} p &= \frac{15}{2}a_2 - \frac{25}{12}(a_1 + 3b_1)^2 \\ q &= -\frac{45}{2}a_3 + \frac{75}{12}a_2(a_1 + 3b_1) - \frac{125}{108}(a_1 + 3b_1)^3 \end{aligned}\right\} \quad (40)$$

then at least one and possibly two or three sets of real positive values can exist.

To summarise, if the inequalities (38) and either (39) or (40) are satisfied then at least one positive real set of values exist for which the system of (2) can be simulated with the circuit of Fig. 4. The network component values may be obtained by solving the cubic of (37) for T_8. Substitution of real positive T_8, so obtained, into (32) through (35) gives on solution a, T_5 and T_7. Choosing arbitrarily a convenient value for one of the capacitors and on solving (30) and (35), the remaining component values may be easily obtained.

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