A NOTE ON ATTRIBUTE LIFE-TESTING

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We shall consider an attribute life-testing problem with replacement. The term 'attribute life-testing' refers to situations where the actual life of the failed items are not known but only the number of failures in given interval of time is available. In earlier papers[1-3] life-testing problems of such nature for the non-replacement case have been treated. As discussed by Bhattacharya[3], we assume that N_i items were installed in a particular department designated by i, out of which n_i has failed within a period T_i , i varying from 1, 2,.............k. We further assume that an item, as soon as it fails is immediately replaced by a new one. This is a more realistic situation because in practice a failed item has to be replaced. It is assumed that the new items come from the same original population with life-time distribution specified by the probability density function $f(t, \theta)$, where $t\ (> o)$ denotes the life-time of an item measured from the origin and θ is single or vector valued parameter to be estimated from the available information.

Consider a particular item from this population which is placed on a life test at t=0. As soon as it fails it is immediately replaced by a new item from the same population, this process being continued for successive failures. We wish to calculate the probability $P_s(T)$ of exactly S failures for this item after a time T. If we suppose that the S failures have taken place at times $t_1, t_2, t_3 \ldots\ldots\ldots, t_s$ respectively, such that $o \leqslant t_1, \leqslant {}_2 \leqslant \ldots\ldots \leqslant t_s \leqslant T$, then the time between each renewal and next failure is governed by the same probability law $f(t, \theta)$. Hence under the assumption that successive failures are statistically independent, we have

$$P_s(T) = \int\int \ldots\ldots \int\int f(t_1)\, f(t_2 - t_1) \ldots\ldots f(t_s - t_{s-1})\, F_c(T - t_s)\, dt_1 \ldots\ldots dt_s , \tag{1}$$

where $f(t)$ has been used instead of $f(t, \theta)$ for brevity,

$$F_c(t) = 1 - \int_o^t f(x)\, dx,$$

and the integration is to be carried over the region

$$o \leqslant t_1 \leqslant t_2 \leqslant \ldots\ldots\ldots \leqslant t_s \leqslant T$$

Let us evaluate this integral when the probability law of life-time is a gamma distribution given by

$$f(t) = \frac{t^{p-1}\ Exp\{-t/\alpha\}}{\alpha^p\, \Gamma(p)}, \quad t > o,\ p > o,\ \alpha > o, \tag{2}$$

So that

$$F_c(t) = \frac{\Gamma(p, t/\alpha)}{\Gamma(p)}, \tag{3}$$

where $\Gamma(p\,x) = \int_x^\infty e^{-t}\, t^{p-1}\, dt.$

We shall also use the notations

$$\gamma(p, x) = \Gamma(p) - \Gamma(p, x),$$

and $$I_p(x) = \frac{\gamma(p, x)}{\Gamma(p)}$$

The integral given by (1) is merely a S-fold convolution $f * f * \ldots * f * F_c$, where f appears S times and

$$g * h = \int_0^x g(x-t)\, h(t)\, dt$$

Since the Laplace transforms multiply under convolution, we get

$$\int_0^\infty e^{-ZT} P_s(T)\, dT = \left\{ \int_0^\infty e^{-ZT} f(t)\, dt \right\}^S \cdot \left\{ \int_0^\infty e^{-ZT} F_c(t)\, dt \right\}, \text{ where} \tag{4}$$

the existence of the transforms holds good under usual conditions. The first integral on the right hand side of (4) is found to be $(1+\alpha z)^{-p}$, whereas for the second member, after substituting for $F_c(t)$ from (3), the Laplace transform turns out to be (c.f. [4], formula 30, pp. 178).

$$Z^{-1}\left[1 - (1 + \alpha Z)^{-p} \right], \text{ provided } p > -1$$

and $$Re\, Z > -Re\, \alpha^{-1}$$

Hence $$\int_0^\infty e^{-ZT} P_s(T)\, dT = Z^{-1}(1+\alpha Z)^{-sp}\left[1 - (1 + \alpha Z)^{-p} \right]$$

We now simply apply the Complex inversion for the Laplace transforms to get (*c.f.*[4], formula 3, pp. 238)

$$P_s(T) = \frac{\gamma(sp\,T/\alpha)}{\Gamma(sp)} - \frac{\gamma(\overline{s+1}\,.\,p,\,T/\alpha)}{\Gamma(\overline{s+1}\,.\,p)}$$

$$= I_{sp}(T/\alpha) - I_{(s+1)p}(T/\alpha)$$

Since this probability involves the incomplete gamma function ratio, it is futile to make an attempt for a direct solution of the maximum likelihood equations for p and α. However the case p=1 leads to a simple straightforward solution. In this case the life-time distribution is exponential which appears very frequently in the field of life-testing. For p=1, P_s (T) reduces to

$\frac{T^s}{S!\alpha^s}$ $Exp\,\{-T/\alpha\}$, so that the probability ${}_NP_n\,(T)$ of exactly n failures in time T for the replacement case, if N items are simultaneously placed on test, is easily seen, by considering the generating function of $P_s\,(T)$, to be $\frac{1}{n!}\left(\frac{NT}{\alpha}\right)^n \;Exp\left\{-\frac{NT}{\alpha}\right\}$, which is even otherwise a direct consequence of Poisson's distribution. Using this result, the likelihood function of observations specified in the very beginning is

$$l = \prod_{i=1}^{K} \frac{1}{n_i\,!}\left(\frac{N_i\,T_i}{\alpha}\right)^{n_i} Exp\left\{-\frac{N_i\,T_i}{\alpha}\right\},\text{ so}$$

that $\frac{d\log l}{d\alpha} = 0$ gives $\hat{\alpha} = \frac{\sum_{i=1}^{k} N_i\,T_i}{\sum_{i=1}^{k} n_i}$, which is an estimate of the average life as discussed earlier[1].

It can also be easily seen that

$$E\left\{-\frac{d^2\log l}{d\alpha^2}\right\} = \sum_{i=1}^{k}\frac{N_i\,T_i}{\alpha^3},\text{ so}$$

that the asymptotic variance of the estimate is

$V(\hat{\alpha}) = \frac{\alpha^3}{(\sum_{i=1}^{k} N_i\,T_i)}$. Hence an estimate for the standard error of the estimate $\hat{\alpha}$ may be calculated from

$$\frac{\left(\sum_{i=1}^{k} N_i\,T_i\right)}{\left(\sum_{i=1}^{k} n_i\right)^{\frac{3}{2}}}$$

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