THE CARRYING CAPACITY OF A CLAMPED SPHERICAL CAP UNDER ROTATIONALLY SYMMETRIC LOADING

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The problem of a clamped spherical cap with a built-in-edge subjected to a rotationally symmetric pressure has been considered. The material of the shell is rigid—perfectly plastic and obeys the generalized square yield condition. The carrying capacity of the cap has been determined when (*i*) a uniform pressure is applied over the surface and (*ii*) a concentrated load is applied at the vertex of the cap.

The plastic analysis of rotationally symmetric shells is of fairly recent origin. The yield surface for a shell whose material obeys the Tresca yield condition was derived by Onat and Prager[1]. The corresponding equations for a shell material obeying the Mises' yield condition were obtained by Hodge[2]. In addition, two linear approximations have been proposed by Hodge[3,4]. More recently, a much simpler yield condition for a general rotationally symmetric shell has been proposed by the author[5].

In this paper the problem of a clamped spherical cap subjected to normal pressure is considered. The material of the shell is assumed to obey the generalized square yield condition[5].

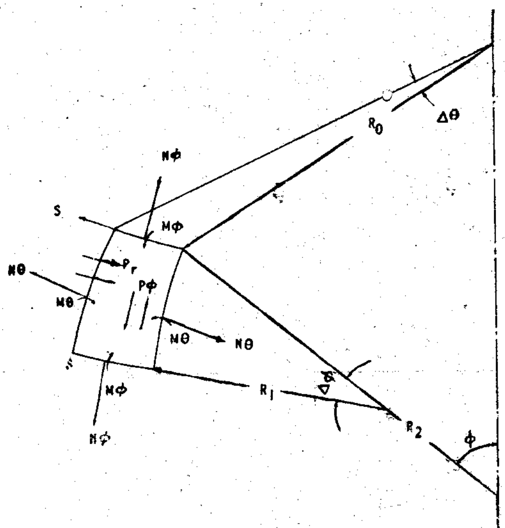

Fig. 1—Element of shell of revolution

BASIC EQUATIONS

Fig. 1 shows a rotationally symmetric shell element whose state of stress is described by four generalized stresses: the circumferential and meridional bending moments M_θ and M_ϕ and the circumferential and meridional membrane forces N_θ and N_ϕ. The shear force S is not considered to be a generalized stress but has the nature of a reaction. The load per unit area of the middle surface of the shell has the components P_ϕ in the direction of the meridian and P_r in the normal direction. The element has the distance R_o from the axis of revolution and its principal radii of curvature are R_1 and R_2. The generalized stresses must satisfy three equations of equilibrium which may be written[6]

$$\left.\begin{aligned} (r_o n_\phi)' - r_1 n_\theta \cos\phi - r_o s + r_o r_1 p_\phi &= 0 \\ r_o n_\phi + r_1 n_\theta \sin\phi + (r_o s)' + r_o r_1 p_r &= 0 \\ k[(r_o m_\phi)' - r_1 m_\theta \cos\phi] - r_o r_1 s &= 0 \end{aligned}\right\} \quad (1)$$

where we have defined $n = N/N_o = N/2\sigma_o H$, $m = M/M_o = M/\sigma_o H^2$, $s = S/N_o$, $p_\phi = LP_\phi/N_o$, $p_r = LP_r/N_o$, $r = R/L$; σ_o is the tensile yield stress of the material. The shell is of uniform thickness 2H and radius R and L is a typical length. Primes denote differentiation with respect to ϕ.

The state of strain is described by four generalized strains which may, in turn, be expressed in terms of the meridional and normal components of the displacement V and W. The generalized strain rates and the velocities are related by

$$\left.\begin{aligned} \dot\epsilon_\theta &= \frac{1}{r_2}(\dot v \cot\phi - \dot w), \quad \dot\epsilon_\phi = \frac{1}{r_1}(\dot v' - \dot w) \\ \dot x_\theta &= -\frac{k \cot\phi}{r_1 r_2}(\dot v + \dot w'), \quad \dot x_\phi = -\frac{k}{r_1}\left(\frac{\dot v + \dot w'}{r_1}\right)' \end{aligned}\right\} \quad (2)$$

The plastic yield condition for a rotationally symmetric shell based on the Tresca criterion has been described by Onat and Prager[1]. Hodge[2] has derived the yield surface based on the Mises' criterion. More recently a linear yield condition for rotationally symmetric shells has been proposed by the author[5]. This yield condition is a linear surface in four-dimensional stress space defined by the eight hyper planes

$$\max\left[\,|n_\theta|, |n_\phi|, |m_\theta|, |m_\phi|\,\right] \leqslant 1 \quad (3)$$

Two two-dimensional projections of the four-dimensional yield surface are illustrated in Fig. 2.

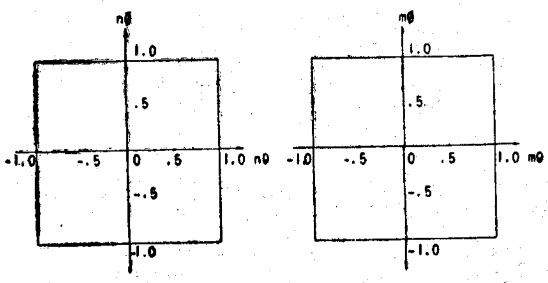

Fig. 2—Generalised square yield condition

The plastic potential flow rule states that if the stress point is in the interior of the yield surface (3), then the strain-rate vector

$$E\,(\dot{\epsilon}_\theta, \dot{\epsilon}_\phi, \dot{x}_\theta, \dot{x}_\phi) = 0 \quad ;$$

if the stress point is in contact with one of the hyper-planes of the yield surface, then E is directed along the outward normal of that hyperplane; if the stress point is at the intersection of two or more hyperplanes, then E must be a linear combination with non-negative coefficients of the outward normals to the hyperplanes involved; for the considered rigid perfectly plastic material, the stress point can never be outside the yield surface.

Finally, the boundary conditions of the problem must be stated. At the centre of the shell, isotropy and symmetry demand that

$$\phi = 0 : n_\phi = n_\theta \,,\, m_\phi = m_\theta \,,\, s = 0 \tag{4}$$

(either $\dot{w}' = \dot{v} = 0$ or there is a hinge circle)

At the clamped edge

$$\phi = \alpha : \dot{w} = 0 \tag{5}$$

(either $\dot{w}' = \dot{v} = 0$ or there is a hinge circle)

A hinge circle is a circle across which $\dot{w}'$ and/or $\dot{v}$ are discontinuous. A discontinuity in $\dot{w}'$ is possible if $|\,m_\phi\,| = 1$ and a discontinuity in $\dot{v}$ is possible if $|\,n_\phi\,| = 1$.

CLAMPED SPHERICAL CAP UNDER UNIFORM PRESSURE

If the typical length of the shell is taken to be its radius, the dimensionless radii of the sphere are

$$r_1 = r_2 = 1, \; r_o = Sin\,\phi \tag{6}$$

Fig. 3 shows a clamped spherical cap under uniform pressure. In view of Eq. (6), the equations of equilibrium (1) become

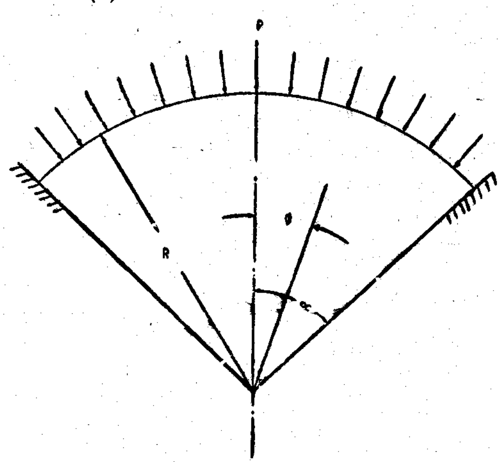

Fig 3—Spherical cap under uniform pressure

$$
\begin{aligned}
&(n_\phi \sin \phi)' - n_\theta \cos \phi = s \sin \phi \\
&(s \sin \phi)' + (\sin \phi)(p + n_\theta + n_\phi) = 0 \\
&k\,[(m_\phi \sin \phi)' - m_\theta \cos \phi] = s \sin \phi
\end{aligned} \tag{7}
$$

The relations (2) between the velocities and strain rates become

$$
\begin{aligned}
&\dot{\epsilon}_\theta = \dot{v} \cot \phi - \dot{w}_\phi, \quad \dot{\epsilon}_\phi = \dot{v}' - \dot{w} \\
&\dot{x}_\theta = -k \cot \phi \,(\dot{v} + \dot{w}'), \quad \dot{x}_\phi = -k(\dot{v} + \dot{w}')'
\end{aligned} \tag{8}
$$

The general procedure for finding the collapse load of a structure is first to make a hypothesis for the stress profile. For any such hypothesis the equilibrium equations and flow rule are solved. The resulting stress and velocity fields must then be examined. The solution will be statically admissible if the stress profile lies everywhere on or inside the yield surface. It will be kinematically admissible if the strain rate vector is directed outward to the yield surface and lies between the appropriate limits at a corner. Solutions which are statically or kinematically admissible will provide lower or upper bounds respectively, while the actual solution is distinguished by the fact that it must be both statically and kinematically admissible.

For the simply supported cap the stress profile was found to correspond to regime 45 of the generalized square yield condition[5]. Hence for the portion of the clamped cap near the vertex, we can anticipate the same stress profile to remain valid. As a first hypothesis, let us assume that the entire cap corresponds to regime 45. This implies that $n_\phi = -1$ and $m_\theta = 1$. The substitution of the above stress relations into the equations of Equilibrium (7) and the use of the boundary conditions (4) result in

$$
\left.
\begin{aligned}
&n_\phi = -1, \quad m_\theta = 1 \\
&n_\theta = -\tfrac{1}{2}\,[p - (p-2)\sec^2\phi] \\
&m_\phi = 1 - \frac{(p-2)}{2k}\left[\frac{1}{\sin\phi}\log(\sec\phi + \tan\phi) - 1\right]
\end{aligned}
\right\} \tag{9}
$$

Since the stress profile is assumed to be everywhere on regime 45, it follows from Equations (8) that the velocity equations are

$$
\left.
\begin{aligned}
&\dot{\epsilon}_\theta = \dot{v}\cot\phi - \dot{w} = 0, \quad \dot{\epsilon}_\phi = \dot{v}' - \dot{w} = -\mu_4, \\
&\dot{x}_\theta = -k\cot\phi\,(\dot{v} + \dot{w}') = \mu_5, \quad \dot{x}_\phi = -k(\dot{v} + \dot{w}')' = 0
\end{aligned}
\right] \tag{10}
$$

where μ_4 and μ_5 are arbitrary positive multipliers. The solution of Equations (10) which satisfies the boundary condition (5) is given by

$$
\begin{aligned}
&\dot{w} = \dot{w}_\circ \cos\phi \left[1 - \frac{\log(\sec\phi + \tan\phi)}{\log(\sec\alpha + \tan\alpha)}\right] \\
&\dot{v} = \dot{w}_\circ \sin\phi \left[1 - \frac{\log(\sec\phi + \tan\phi)}{\log(\sec\alpha + \tan\alpha)}\right] \\
&\mu_4 = \frac{\dot{w}_\circ \tan\phi}{\log(\sec\alpha + \tan\alpha)}, \quad \mu_5 = \frac{k\dot{w}_\circ \cot\phi}{\log(\sec\alpha + \tan\alpha)}
\end{aligned} \tag{11}
$$

where $\dot{w}_o$ is the velocity at the centre of the shell. At $\phi = \alpha$, $\dot{v} = 0$ and hence the tangential displacement is continuous. However equations (11) shows that $\dot{w}$ has a discontinuity at $\phi = \alpha$, which implies that there is a hinge circle at the clamped edge. As stated previously, this discontinuity is possible if $m\phi = -1$ at $\phi = \alpha$. If the stress solution given by Equations (9) should satisfy this condition, the collapse load must be

$$p = 2 + (4k \sin \alpha) [\log (\sec \alpha + \tan \alpha) - \sin \alpha]^{-1} \quad (12)$$

The solution given by equations (9) and (12) will be statically admissible if the stress profile remains on finite faces 4 and 5. This means that the stresses must satisfy

$$-1 \leqslant n_\theta \leqslant 1; \quad -1 \leqslant m_\phi \leqslant 1 \quad (13)$$

The discussion of the above inequalities is facilitated by the following expansion

$$\log (\sec \alpha + \tan \alpha) - \sin \alpha = \tfrac{1}{3} \sin^3 \alpha + \frac{1}{5} \sin^5 \alpha + \ldots$$

It follows from Equation (12) that p is always greater than 2. Equations (9) then show that n_θ is a monotonically increasing function of ϕ, whereas m_ϕ is monotonically decreasing. Hence the above solution will be statically admissible provided that $n_\theta (\alpha) \leqslant 0$. This condition leads to

$$\frac{\cos^2 \alpha}{\sin^3 \alpha} \left[\log \frac{1 + \sin \alpha}{\cos \alpha} - \sin \alpha \right] - 2k \geqslant 0 \quad (14)$$

It is clear from Equations (11) that μ_4 and μ_5 are always positive and hence the velocity solution is kinematically admissible. Inequality (14) is a restriction on the parameters of the cap. For values of the parameters violating inequality (14), some other hypothesis must be made for the stress profile. The solution given in this paper is restricted to values of α and k which are consistent with the assumed stress profile. Fig. 4 shows the collapse pressure given by Equations (12) as a function of α for $k = 0 \cdot 02$.

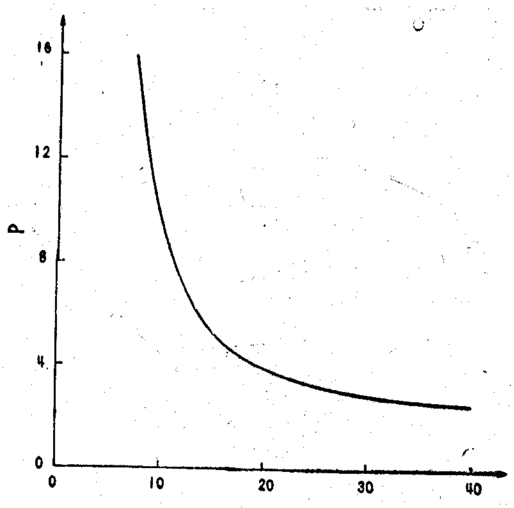

Fig. 4—Collapse pressure of a spherical cap (k—0·02)

CAP UNDER CONCENTRATED VERTEXLOAD

It is not possible to make a direct formal analysis of a shell subjected to a concentrated load since the resulting shear force at the point of load application is infinite. Hence we consider a shell where the load is applied over a small but finite area and to get the "Solution for concentrated load" we pass to the limit as the loaded area tends to zero.

Fig. 5 shows a clamped spherical cap loaded over a portion of the surface. The conditions of equilibrium for the loaded portion of the shell are given by Equations (7). For the portion of the shell on which there is no pressure acting, the corresponding equations are

$$\left.\begin{aligned} &(n_\phi \sin\phi)' - n_\theta \cos\phi = s \sin\phi \\ &(s \sin\phi)' + (\sin\phi)\,(n_\theta + n_\phi), = 0 \\ &k[(m\,\phi \sin\phi)' - m_\phi \cos\phi] = s \sin\phi \end{aligned}\right\} \quad (15)$$

The relations between the velocities and the strain rates are given by Equations (3) and the boundary conditions are given by Equations (4) and (5). In addition the following continuity condition must be satisfied.

At $\phi = \beta$, m_ϕ, n_ϕ, s, $\dot{v}$, $\dot{w}$ and $\dot{w}'$ must be continuous (16)

As already stated when the load was applied over the entire surface of the cap the stress profile was found to correspond to the hyperplanes $n_\phi = -1$ and, $m_\theta = 1$. Hence for values of β slightly less then α, it is reasonable to expect the same stress profile to be valid. The substitution of the above equations of the hyperplanes into the Equations of equilibrium (7) and (15) and the boundary and continuity conditions (4), (5) and (16) result in

$$\begin{aligned} 0 \leqslant \phi \leqslant \beta:\ n &= -1,\ m_\phi = 1 \\ n_\theta &= -\tfrac{1}{2}[p - (p-2)\,Sec^2\,\phi] \end{aligned} \quad (17)$$

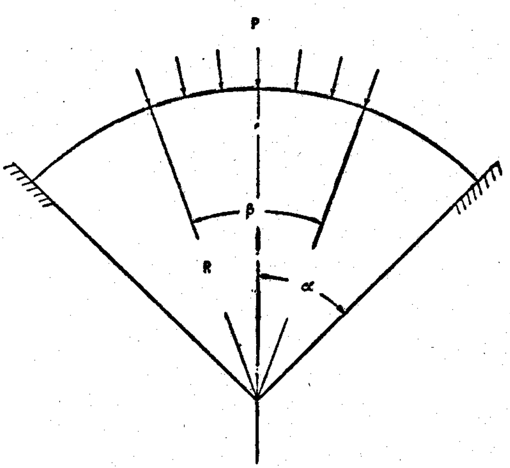

Fig 5—Spherical cap loaded over a portion of the surface.

$$m_\phi = 1 - \frac{(p-2)}{2k}\left[\frac{1}{\sin\phi}\log(\sec\phi + \tan\phi) - 1\right]$$

$$\beta \leqslant \phi \leqslant \alpha : n_\phi = -1,\ m_\phi = 1$$

$$n_\theta = -\frac{Cos^2\beta}{2}\left[p - (p-2)\sec^2\beta\right]\sec^2\phi \tag{18}$$

$$m_\phi = \left(1 - 2\frac{\sin\alpha}{\sin\phi}\right) - \frac{1}{k}\left(1 - \frac{\sin\alpha}{\sin\phi}\right) - \frac{[p\cos^2\beta - (p-2)]}{2k\sin\phi}\log\frac{\sec\alpha + \tan\alpha}{\sec\phi + \tan\phi}$$

Using the condition that m_ϕ must be continuous at $\phi = \beta$, the collapse pressure p is found to be

$$p = \frac{2\{[\log(\sec\alpha + \tan\alpha), - \sin\alpha] + 2k\sin\alpha\}}{\left[\log(\sec\alpha + \tan\alpha) - \sin\beta - (\cos^2\beta)\log\frac{\sec\alpha + \tan\alpha}{\sec\beta + \tan\beta}\right]} \tag{19}$$

The solution given by Equations (17), (18) and (19) will be statically admissible provided inequalities (13) are satisfied. As regards the range $0 \leqslant \phi \leqslant \beta$, it can be verified that the four inequalities (13) are satisfied provided

$$p \leqslant 2 + \frac{4k\sin\beta}{\log(\sec\beta + \tan\beta) - \sin\beta} \tag{20}$$

For the range $\beta \leqslant \phi \leqslant \alpha$, it is found that the inequalities on n_θ are satisfied provided p satisfies the following two inequalities

$$2\sin^2\alpha \leqslant p\sin^2\beta \leqslant 2 \tag{21}$$

However, the complexity of the expression (18) for m_ϕ precludes a general analytical discussion and hence m_ϕ was numerically evaluated and the corresponding inequalities verified. The range of validity of the solution Equations (17), (18) and (19) is indicated as region I in Fig. 6. Since for the above region of validity, the stress profile is everywhere on regime 45, the velocity equations (11) remain valid for this case also.

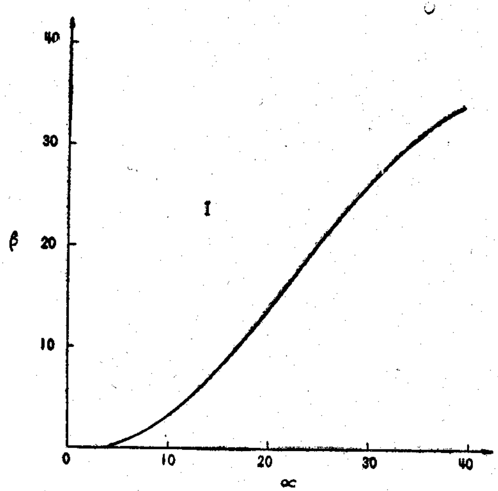

Fig. 6—Range of validity of the solution

The solution for the concentrated load is obtained by passing to the limit as the loaded area tends to zero. Since the collapse pressure is given by Equations (19), the magnitude of the load at collapse is obtained by integrating Equations (19) over the loaded area. Hence the collapse load F is given by

$$F = \int_{0}^{\beta} 2\pi R^2 \, p \, sin \, \phi \, d \, \phi \qquad (22)$$

Substituting the expression for p from Equations (19) into Equations (22) we get

$$F = \frac{4\pi R N_{\circ} [log \, (sec \, \alpha + tan \, \alpha) - (1 - 2k) \; sin \; \alpha](1 - cos \, \beta)}{\left[log \, (sec \, \alpha + tan \, \alpha) - sin \, \beta - (cos^2 \, \beta) \, log \, \dfrac{sec \, \alpha + tan \, \alpha}{sec \, \beta + tan \, \beta} \right]} \qquad (23)$$

By passing to the limit as β tends to zero the concentrated collapse load is found to be

$$F_c = 2\pi R N_{\circ} \left\{ 1 - (1 - 2k) \, (sin \, \alpha)[log \, (sec \, \alpha + tan \, \alpha)]^{-1} \right\} \qquad (24)$$

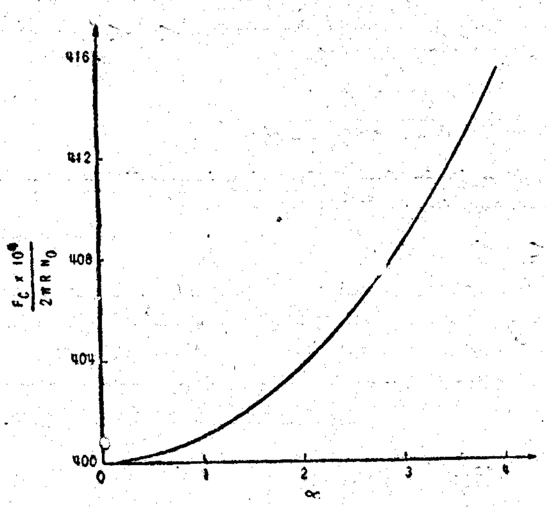

Fig. 7—Concentrated collapse load of spherical cap (k—0·02)

The velocity field given by Equations (11) remains unchanged as β tends to zero. Fig. 7 shows the collapse load ($F_c/2\pi R N_{\circ}$) versus the angle of the cap α, for a particular value of k.

REFERENCES

1. Onat, E.T., and Prager, W., *Proc., Roy, Netherlands, Acad. Sci.* Series B, **57** (1954), 534.

2. Hodge, P.G., Jr., *Quart. Appl. Math.* **18** (1961), 305.

3. Hodge, P.G., Jr., "The Linearization of Plasticity Problems by means of Non-homogeneous Materials" *Proc., Symp. on Non-Homogeneity in Elasticity and Plasticity*, Warsaw, (Pergamon Press), 1958.

4. Hodge, P.G. Jr., *J. App. Mech.*, **27** (1960), 323.

5. Sankaranarayanan, R., "A Generalized Square Yield Condition for shells of Revolution", *Proc. Indian Acad. Sci.* (In Press).

6. Hodge, P.G. Jr., "Plastic Analysis of Structures", (McGraw-Hill Book Company, Inc., New York), 1959.