ON HYDROMAGNETIC POROUS-WALL COUETTE TYPE FLOW II

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This paper reports the results of a study of hydrodynamic flow through a porous walled annular channel under the assumption that the amount of fluid that enters the annular space through the inner tube is equal to the amount that flows out of the outer tube.

The results of an investigation on the effect of a uniform external transverse magnetic field on the geometry of velocity profiles, separation and skin friction of steady-state incompressible viscous couette-type flow of an infinitely electrically conducting fluid between two equally porous parallel planes have been reported in an earlier paper[1]. The investigation has now been extended to the flow through a porous-walled annular channel under the assumption that the amount of fluid that enters the annular space through the inner tube is equal to the amount that flows out of the outer tube. In the absence of magnetic field, couette-type flow of an incompressible viscous fluid through a porous-walled annulus has already been studied by the author[2] and the results obtained were similar to those obtained by Lilley[3] for couette-type flow between two equally porous parallel planes. The results obtained in this paper, though qualitatively similar, differ quantitatively from those reported earlier[1] due to change in channel geometry.

BASIC EQUATIONS

We consider steady state laminar flow of an incompressible viscous electrically conducting fluid through a porous-walled annulus in the presence of an external, uniform and radially transverse magnetic field $\bar{H}_0$. The outer tube $r=a$ is taken moving with a uniform velocity U in the axial (z) direction while the inner tube $r=b$ is stationary. We use cylindrical polar coordinates (r, φ, z) with the central axis of the annulus as z-axis. Assuming that,

(*i*) electrical conductivity σ of the fluid is large enough to ignore displacement currents,

(*ii*) there is no φ - component of velocity and magnetic field,

(*iii*) radial and axial components of velocity and magnetic field depend on r alone,

(*iv*) no electric field is present, and that

$$h = \frac{\bar{H}}{\sqrt{4\pi\rho}}$$

(h has dimensions of velocity; ρ is density of the fluid), the divergence relations of velocity and magnetic field give,

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$$V_r = \frac{\lambda}{\eta} V_{\circ} \qquad (1)$$

and

$$h_r = \frac{h_{\circ}}{\eta} \qquad (2)$$

where

$\lambda = \dfrac{b}{a}$ is annulus radii ratio,

$\eta = \dfrac{r}{a}$ is non-dimensional radial distance parameter which takes the values λ and 1 at the inner and outer tubes respectively,

$V_{\circ}$ is the velocity with which the fluid is injected into the annular space through the inner tube.

$h_r = h_{\circ}$ at $\eta = 1$ as stipulated in the statement of the problem vide (2).

Introducing non-dimensional quantities by writing,

$V = \dfrac{V_z}{U}$, $\qquad H = \dfrac{h_z}{U}$ etc.

$\xi = \dfrac{z}{a}$, $\qquad \varpi = \dfrac{p}{\frac{1}{2}\rho U^2}$, p being pressure.

and making use of the assumptions and (1) & (2), the equations of hydromagnetics[4] (in e.m.u) reduce to,

$$\frac{dH^2}{d\eta} + \frac{\partial \varpi}{\partial \eta} - 2\lambda^2 \frac{V_{\circ}^2}{\eta^3} = O \qquad (3)$$

$$\eta \frac{d^2 V}{d\eta^2} + \frac{dV}{d\eta} - R \frac{dV}{d\eta} + \frac{R_M}{\epsilon} \frac{dH}{d\eta} + C\eta = O \qquad (4)$$

$$\epsilon R_H - R_M \; V = \eta \frac{dH}{d\eta} \qquad (5)$$

where

$R = \dfrac{bV_{\circ}}{\nu}$, cross-flow Reynolds number (γ being kinematic viscosity),

$R_M = \dfrac{ah_{\circ}}{\nu_H}$, magnetic Reynolds number $\nu_H = \dfrac{1}{4\pi a \mu}$ being magnetic viscosity; μ is magnetic permeability of the fluid),

$\epsilon = \dfrac{\nu}{\nu_H}$, a non-dimensional parameter characteristic of fluid under consideration. For Mercury under ordinary conditions $\epsilon = \dfrac{10^{-6}}{6}$,

$$R_e = \frac{aU}{\nu}\,,\quad \text{Couette Reynolds number,}$$

and $C = -\dfrac{R_e}{2}\dfrac{\partial \varpi}{\partial \xi}$ is a term which refers to couette-flow problem with super-imposed axial pressure gradient.

We take the inner tube to be electrically conducting and the outer one to be non-conducting so that axial component of magnetic field H will be continuous across the outer tube only. Hence the boundary conditions to be satisfied by V and H are

$$\left.\begin{aligned} V &= 0\,, & \eta &= \lambda \\ V &= 1\,, & \eta &= 1 \\ H &= 0\,, & \eta &= 1 \end{aligned}\right\} \tag{6}$$

(3) to (5) along with the boundary conditions (6) comprise the basic equations of the problem.

SOLUTION OF THE PROBLEM

Eliminating V between (4) and (5), the differential equation determining H comes out to be

$$\eta^3 \frac{d^3H}{d\eta^3} + \left\{3 - R(1+\epsilon)\right\}\eta^2 \frac{d^2H}{d\eta^2} + \left\{1 - R(1+\epsilon) + \epsilon R^2 - M^2\right\}\eta \frac{dH}{d\eta} = C R_M \eta^2 \tag{7}$$

where $M = \dfrac{a h_0}{\sqrt{\nu \nu_H}}$ is Hartmann number.

On using the transformation $\eta = e^{\zeta}$ the above differential equation transforms to

$$\frac{d^3H}{d\zeta^3} - R(1+\epsilon)\frac{d^2H}{d\zeta^2} + (\epsilon R^2 - M^2)\frac{dh}{d\zeta} = C R_M e^{2\zeta} \tag{8}$$

Its solution in terms of original independent variable η is,

$$H = R_M (k_1 + k_2\, \eta^{\alpha} + k_3\, \eta^{\beta} + k\, \eta^2) \tag{9}$$

provided α (or β) $\neq 2$

in which k_1, k_2 and k_3 are constants of integration and

$$\alpha,\ \beta = \frac{1}{2}\left\{ R(1+\epsilon) \pm \sqrt{R^2(1-\epsilon)^2 + 4M^2}\right\} \tag{10}$$

$$k = \frac{C}{2(\alpha-2)\ (\beta-2)}$$

Substituting for H from (9) into (5) the expression for V comes out to be

$$V = \epsilon R k_1 + (\epsilon R - \alpha) k_2\, \eta^{\alpha} + (\epsilon R - \beta) k_3\, \eta^{\beta} + k(\epsilon R - 2)\eta^2 \tag{11}$$

provided α(or β) $\neq 2$

The constants of integration k_1, k_2 and k_3 are given, on using the boundary conditions (6), by

$$\left.\begin{aligned}
k_1 &= -(k_2 + k_3 + k) \\
k_2 &= \frac{\left\{1 - k(1-\lambda^2)(\epsilon R - 2)\right\}\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - k\left\{2\lambda^2 + \epsilon R(1-\lambda^2)\right\}(\epsilon R - \beta)(1-\lambda^{\beta})}{(\epsilon R - \alpha)(1-\lambda^{\alpha})\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - (\epsilon R - \beta)(1-\lambda^{\beta})\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\}} \\
k_3 &= \frac{\left\{1 - k(1-\lambda^2)(\epsilon R - 2)\right\}\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\} - k\left\{2\lambda^2 + \epsilon R(1-\lambda^2)\right\}(\epsilon R - \alpha)(1-\lambda^{\alpha})}{(\epsilon R - \alpha)(1-\lambda^{\alpha})\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - (\epsilon R - \beta)(1-\lambda^{\beta})\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\}}
\end{aligned}\right\} \quad (12)$$

In the absence of magnetic field ($M = 0$, α, $\beta = R$, ϵR) the above solution reduces to the one already obtained[2].

It should be noticed that the solution given by (9), (11) and (12) does not hold when either α or β has the value 2. This special case is dealt with in the last section.

Having found H as given by (9) and (12) the pressure distribution in the flow region is yielded, on integrating (3) by

$$\varpi(\xi, \eta) + H^2 + \frac{\lambda^2}{\eta^2} V_c^2 = -\frac{2C}{R_e} \cdot \xi \quad (13)$$

whence

$$\varpi(0, \eta) - \varpi(\xi, \eta) = -\frac{2C}{R_e} \cdot \xi \quad (14)$$

Thus the axial pressure-drop varies linearly with the main flow direction.

SEPARATION

The separation at the fixed tube $\eta = \lambda$ is obtained when

$$\left[\frac{dV}{d\eta}\right]_{\eta = \lambda} = 0 \quad (15)$$

which, on using the expression (11) for V, is equivalent to

$$\alpha(\epsilon R - \alpha)k_2 \lambda^{\alpha - 2} + \beta(\epsilon R - \beta)k_3 \lambda^{\beta - 2} + 2k(\epsilon R - 2) = 0 \quad (16)$$

Solving (16) for C under the approximation $\epsilon << 1$, we get,

$$C = \frac{(\alpha' - 2)(\beta' - 2)(\alpha' - \beta')\lambda^{R-2}}{\alpha'(\lambda^{\alpha'} - \lambda^{R-2}) - \beta'(\lambda^{\beta'} - \lambda^{R-2}) + 2(\lambda^{\beta'} - \lambda^{\alpha'})} \quad (17)$$

where

$$\alpha', \beta' = \frac{1}{2}\left\{R \pm \sqrt{R^2 + 4M^2}\right\}$$

In the absence of magnetic field ($M=0$, $\alpha'=R$, $\beta'=0$) the expression (17) for C reduces to

$$C = \frac{2R\,(R-2)\,\lambda^{R-2}}{R(1-\lambda^2)\lambda^{R-2} - 2\,(1-\lambda^{R})} \tag{18}$$

as in reference 2.

TABLE 1

VALUES OF $\frac{R_e}{2} \cdot \frac{\partial \varpi}{\partial \xi}$ FOR WHICH SEPARATION OCCURS IN ABSENCE ($M=0$) AND PRESENCE ($M=5$) OF MAGNETIC FIELD FOR DIFFERENT VALUES OF R.

$\left(\frac{R_e}{2} \cdot \frac{\partial \omega}{\partial \xi}\right)$ Separation at $\eta = \lambda$

R	Injection		Suction	
	M=0	M=5	M=0	M=5
0	53·760	48·843	53·760	48·843
2	46·350	41·969	61·728	56·247
4	39·506	35·672	70·145	64·135
6	33·278	29·946	78·980	72·550
8	27·668	24·855	88·174	81·319
10	22·729	20·407	97·660	91·954

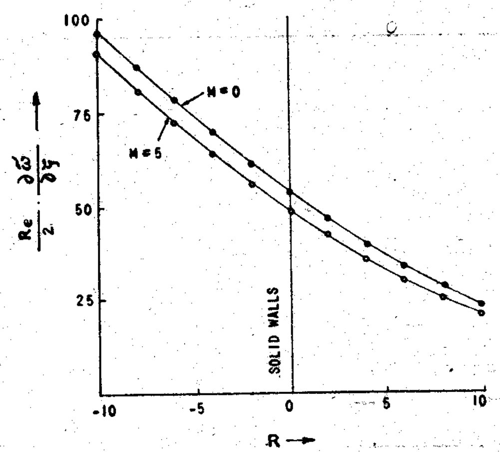

Fig. 1—Cross flow Reynolds number, R plot of $\frac{R_e}{2} \cdot \frac{\partial \bar{\omega}}{\partial \xi}$ (separation at fixed tube $\eta=\lambda$) against R for Hartmann number M=0 and 5.

Fig. 1 gives the plot of (17) for M=0 and 5. It shows that in presence of magnetic field, smaller adverse axial pressure gradient would provoke separation at the fixed tube and that it is lesser in case of injection than that of suction of fluid through the inner tube.

SKIN FRICTION

When the axial pressure gradient is zero (*i.e.* $C=0$), the skin friction experienced by the fixed tube is given by,

$$\frac{\tau}{\lambda U^2} = \frac{1}{R_e}\left[\frac{dV}{d\eta}\right]_{\eta=\lambda} = \frac{1}{R_e}\left\{\alpha(\epsilon R-\alpha)k_{20}\lambda^{\alpha-1} + \beta(\epsilon R-\beta)k_{30}\lambda^{\beta-1}\right\} \tag{19}$$

in which k_{20} and k_{30} are values of k_2 and k_3 when $C=0$

Under the approximation $\epsilon << 1$, the expression (19) for $\frac{\tau}{\rho U^2}$ becomes,

$$\frac{\tau}{\rho U^2} = \frac{1}{R_e}\cdot\frac{(\alpha'-\beta')^{R-1}}{\lambda^{\beta'}-\lambda^{\alpha'}} \tag{20}$$

TABLE 2

VALUES OF SKIN ERICTION IN ABSENCE (M=0) AND PRESENCE (M = 5) OF MAGNETIC FIELD FOR DIFFERENT VALUES OF R.

$$R_e\cdot\frac{\tau}{\rho U^2}$$

	Injection		Suction	
R	M=0	M=5	M=0	M=5
0	5·603	4·589	5·603	4·589
2	4·444	3·643	6·944	5·692
4	3·468	2·849	8·469	6·955
6	2·664	2·194	10·164	8·373
8	2·015	1·666	12·015	9·936
10	1·503	1·249	14·004	11·634

which, in absence of magnetic field, reduces to

$$\frac{\tau}{\rho U^2} = \frac{1}{R_e}\cdot\frac{R\lambda^{R-1}}{1-\lambda^{R}} \tag{21}$$

as in reference 2.

Fig 2, which gives plot of (20) for M=0, 5 shows that skin friction experienced by the fixed tube is reduced due to presence of magnetic field. It further illustrates that skin friction is lesser in case of injection than that of suction of fluid through the inner tube.

Special Case: Either α or $\beta = 2$

We take $\alpha = 2$. This implies a functional relationship between R and M given by,

$$M^2 = 4 - 2R(1+\epsilon) + \epsilon R^2 \tag{22}$$

$$\simeq 4 - 2R \text{ when } |R| \text{ is not very large}$$

Since from (10), $\alpha + \beta = R(1+\epsilon)$, we, therefore, have $\beta = R(1+\epsilon) - 2$.

In this case, the solution of the differential equation (8) in terms of η is given by,

$$H = R_M\,(l_1 + l_2\,\eta^{\alpha} + l_3\,\eta^{\beta} + l\,\eta^2 \ln \eta) \tag{23}$$

l_1, l_2 and l_3 being constants of integration and

$$l = \frac{C}{2(2-\beta)} = \frac{C}{8 - 2R(1+\epsilon)}$$

The expression for velocity is

$$V = \epsilon R l_1 + (\epsilon R - \alpha)\,l_2\,\eta^{\alpha} + (\epsilon R - \beta) l_3\,\eta^{\beta} + l\eta^2\{(\epsilon R - 2)\ln\eta - 1\} \tag{24}$$

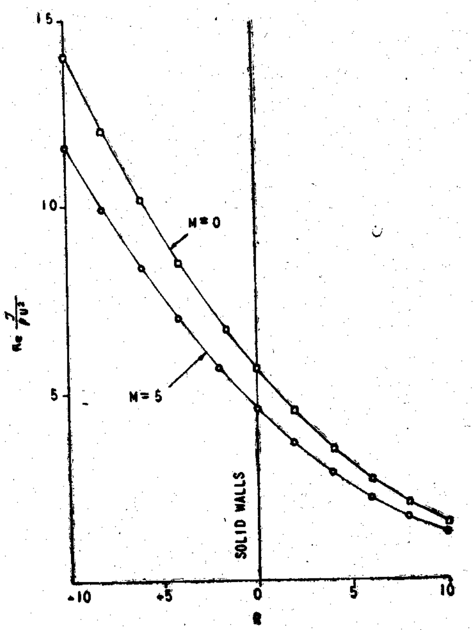

Fig. 2—Plot of $R_e \cdot \dfrac{\tau}{\rho U^2}$ against R for Hartmann Number M = 0 and 5.

On using the boundary conditions (6), l_1, l_2 and l_3 are given by

$$\left.\begin{aligned} l_1 &= -(l_2 + l_3) \\ l_2 &= \frac{(1 + l - L)\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - L(\epsilon R - \beta)(1 - \lambda^{\beta})}{(\epsilon R - \alpha)(1 - \lambda^{\alpha})\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - (\epsilon R - \beta)(1 - \lambda^{\beta})\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\}} \\ l_3 &= -\frac{(1 + l - L)\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\} - L(\epsilon R - \alpha)(1 - \lambda^{\alpha})}{(\epsilon R - \alpha)(1 - \lambda^{\alpha})\left\{(\epsilon R - \beta)\lambda^{\beta} - \epsilon R\right\} - (\epsilon R - \beta)(1 - \lambda^{\beta})\left\{(\epsilon R - \alpha)\lambda^{\alpha} - \epsilon R\right\}} \end{aligned}\right\} \quad (25)$$

where $L = l\{\lambda^2 - (\epsilon R - 2)\lambda^2 ln\, \lambda\}$.

In the absence of magnetic field, this case corresponds to the case $R = 2$ of reference 2 and the expression (24) for V reduces to

$$V = \left(\frac{C}{2}\, ln\, \lambda - 1\right) \frac{\lambda^2}{1 - \lambda^2} + \frac{1 - \frac{C}{2}\lambda^2\, ln\, \lambda}{1 - \lambda^2} . \eta^2 - \frac{C}{2}\, \eta^2\, ln\, \eta \qquad (26)$$

which is the same as in reference 2

Under the approximation $\epsilon << 1$, the expression for skin friction and the value of C for which separation will occur at the fixed tube are given by

$$\frac{\tau}{\rho U^2} = \frac{1}{R_e} \cdot \frac{(4 - R)\lambda^{R-1}}{\lambda^{R-2} - \lambda^2} \qquad (27)$$

and

$$C = \frac{2(4 - R)^2\, \lambda^{R-2}}{4(1 + ln\lambda)\,(\lambda^{R-2} - \lambda^2) - (4 - R)\lambda^{R-2} + \lambda^2(1 + 2ln\lambda)\{2 - (R - 2)\lambda^{R-4}\}} \qquad (28)$$

In the absence of magnetic field, this case corresponds to the special case $R = 2$ of reference 2 and expressions for $\frac{\tau}{\rho U^2}$ and C accordingly reduce to

$$\frac{\tau}{\rho U^2} = \frac{1}{R_e} \cdot \frac{2\lambda}{1 - \lambda^2} \qquad (29)$$

and

$$C = \frac{4}{1 - \lambda^2 + 2ln\lambda} \qquad (30)$$

as in reference 2.

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