ACCEPTANCE SAMPLING BY VARIABLES WITH LIFE-TEST OBJECTIVES

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Sampling plans by variables with special reference to life-test problems based on Weibull distribution are obtained. The life-test is terminated at the *r-th* failure. The sampling plan is such that a lot is accepted with a preassigned (high) probability when the specified mean life can be established. OC functions of the plans are obtained and Producer's risk is also discussed. A method of obtaining minimum sample size has been proposed so as to obtain the required number of failures within a prescribed time limit with a given probability.

Recently some work has been done on acceptance sampling based on life tests[1,2,3,4]. They are all based on the attribute plans. Our object is to discuss some sampling plans by variables with such life-tests objectives based on Weibull distribution. This life distribution which includes exponential distribution as a particular case is one of the statistical models for life-length of materials such as electrical and electronic goods. [See for example, Kao[5] and Weibull[6]].

In many life test problems it is customary to terminate the experiment as soon as first r failures are observed. If n units are placed on life test and if $t_1, t_2, \ldots\ldots, t_r$ be the first r failure times (measured from the start of the test), then assuming a parametric model for life time, we can set confidence limits on one of the parameters (or on a function of this parameter). Alternatively we can obtain a decision procedure which assures a specified mean life with a preassigned (high) probability, thus providing the consumers protection. The parameters model is assumed to be Weibull distribution with its shape parameter p known.

CHARACTERISTICS OF WEIBULL DISTRIBUTION

The probability density is given by

$$f(t) = (p/\theta)\, t^{p-1} \exp\{-t^p/\theta\}, \quad 0 < t < \infty, \quad p > 0, \quad \theta > 0 \tag{1}$$

The moments of the distribution about the origin are

$$\mu^1_r = E(t^r) = \theta^{(r/p)}\, \Gamma[1 + (r/p)], \quad \text{so that}$$

$$\mu^1_1 = \text{Mean} = [\theta/_p]^{1/p}\, \Gamma[1/p] \tag{2}$$

It is easily seen that Weibull distribution corresponds to a failure rate of :

$$\lambda(t) = (p/\theta)\, t^{p-1}$$

We shall give here the maximum likelihood estimate of θ based on the first r ordered observation $t_1, t_2, \ldots\ldots\ldots, t_r$ when n units are placed on the test. The distribution of the above estimate may be obtained on the same lines as given by Sobel[7] for exponential distribution.

Theorem 1—The maximum likelihood estimate of θ is given by

$$\hat{\theta}_{rn} = \left[\{ \sum_{i=1}^{r} t_i^{p} + (n-r)\, t_r^{p} \}/r \right] \qquad (3)$$

which is unbiased and the distribution of $2r\hat{\theta}_{rn}/\theta$ is the x^2 variate with $2r$ degrees of freedom.

Proof—The likelihood function of the sample $t_1, t_2, \ldots\ldots\ldots\ldots, t_r$ is given by

$$L = \left[n\,!/(n-r)\,! \right] \left[\prod_{i=1}^{r} (p/\theta) t_i^{p-1} \right] \exp\left\{ \frac{-\Sigma t_i^{p}}{\theta} \right\} \left[\int_{t_r}^{\infty} (p/\theta)\, t^{p-1} \exp\left\{ \frac{-t^{p}}{\theta} \right\} dt \right]^{n-r} \qquad (4)$$

The logarithm of the liklihood (4), when maximised, gives (3). It may be noted that $p=1$ gives Sobel's results.

To obtain the distribution of $\hat{\theta}_{rn}$ we put

$$y_1 = t_1^{p}, \quad y_i = t_i^{p} - t_{i-1}^{p} \quad (i = 2,3,\ldots\ldots, r).$$

It may be easily seen that the joint distribution of $y_1, y_2 \ldots\ldots\ldots, y_r$ is given by the probability density

$$f(y_1, y_2, \ldots y_r) = \left[\prod_{i=1}^{r} \left((n-i+1)/\theta \right) \exp\left\{ -\frac{(n-i+1)\, y_i}{\theta} \right\} \right] \qquad (5)$$

From (5) it is apparent that $y_1, \ldots\ldots\ldots\ldots y_r$ are independent and the distribution of $(n-i+1)y_i$ is given by the probability density $1/\theta \exp\left[\frac{-x}{\theta} \right]$, which has the characteristic function $E(e^{idx}) = (1 - i\,\alpha\,\theta)^{-1}$. Hence the characteristic function of the distribution of $\hat{\theta}_{rn} = \sum_{i=1}^{r} \frac{(n-i+1)\, y_i}{r}$ is given by

$$\phi_{\hat{\theta}_{rn}}(\alpha) = \left(1 - \frac{\alpha i \theta}{r} \right)^{-r} \qquad (6)$$

By inversion theorem, the probability density is obtained as

$$f(\hat{\theta}_{rn}) = \frac{1}{\Gamma(r)} \left(\frac{r}{\theta} \right)^{r-1} \left(\hat{\theta}_{rn} \right)^{r-1} \exp\left\{ \frac{-r\hat{\theta}_{rn}}{\theta} \right\} \qquad (7)$$

From (7), it may be easily seen that $E(\hat{\theta}_{rn}) = \theta$ and $(2r\hat{\theta}_{rn}/\theta)$ is distributed as χ^2 on $2r$ degrees of freedom.

Remarks—It may be noted that the distribution of $\hat{\theta}_{rn}$ is independent of p and n, and is the same as Sobel's result.

CHOICE OF SAMPLING PLAN

A common practice in life testing is that n units are placed on the test and the life test is terminated as soon as a preassigned number r of failures are observed. This preassigned r generally depends on cost and time considerations. We want a sampling plan which will establish a desired mean life of the lot with a given (high) probability. This can be done with the help of the statistics (3), as follows:

(*i*) n items from the lot are put on the test and the first r failure times $t_1, t_2 \ldots, t_r$ are noted.

(*ii*) compute the statistic $\hat{\theta}_{rn}$, and accept the lot of $\hat{\theta}_{rn} \geqslant A$, otherwise reject.

The choice of the sampling plan is the choice of A, which is made according to the following consideration:

Given a value $\theta_\circ$ and a number P^* $(0<P^*<1)$ we want that the probability of the acceptance of the lot under this plan, whenever $\theta<\theta_\circ$ or equivalently the mean life is less than a specified value, is less than or equal to P^*.

Mathematically,

$$P_r \; \hat{\theta}_{rn} \geqslant A/\theta<\theta_\circ\} \leqslant P^* \tag{8}$$

or $P_r \; \{\chi^2_{2r} \geqslant \frac{2rA}{\theta} /\theta<\theta_\circ\} \leqslant P^*$, using theorem 1, where χ^2_{2r} denotes χ^2 variate with $2r$ degrees of freedom.

This condition is, of course, satisfied if we choose A such that

$$P_r \; \{\chi^2_{2r} \geqslant \frac{2rA}{\theta_\circ} \leqslant P^* \tag{9}$$

Minimum values of $\frac{A}{\theta_\circ}$ for a given value of r, satisfying the inequality (9) for $P^*=0{\cdot}01$, $0{\cdot}05$, $0{\cdot}10$ and $0{\cdot}20$ have been shown in Table 1.

TABLE 1

TABLE SHOWING MINIMUM VALUE OF $\frac{A}{\theta_\circ}$ FOR A GIVEN VALUE OF CONSUMER'S RISK

r / P^*	·01	·05	·10	·20
1	4·605	2·996	2·303	1·610
2	3·319	2·372	1·945	1·497
3	2·802	2·099	1·778	1·426
4	2·511	1·938	1·670	1·379
5	2·321	1·831	1·599	1·344
6	2·184	1·752	1·546	1·318
7	2·082	1·693	1·504	1·297
8	2·000	1·644	1·471	1·279
9	1·934	1·604	1·444	1·264
10	1·878	1·570	1·420	1·252
11	1·831	1·542	1·401	1·241
12	1·791	1·517	1·383	1·231
13	1·771	1·496	1·368	1·223
14	1·724	1·476	1·354	1·215
15	1·696	1·459	1·342	1·208

It is apparent that p does not occur explicitly in the inequality (9) and that the same Table 1 may be used for obtaining A, whatever be the value of p, but the statistic $\hat{\theta}_{rn}$ is different for different values of p. It is also seen that the choice of the sampling plan does not depend on the number of units placed on the test but this number may be fixed according to time considerations as discussed later.

OPERATING CHARACTERISTICS

OC function is the probability of accepting a lot under a given plan and is given by

$$Pr\left\{\hat{\theta}_{rn} \geqslant A\right\} = Pr\left\{\chi^2_{2r} \geqslant \left(\frac{2rA}{\theta_\circ} \Big/ \frac{\theta}{\theta_\circ}\right)\right\} \quad (10)$$

OC curves for a few selected sampling plans (r=5, 10, 15; p^*=0·01, 0·05, 0·1) are drawn in figures 1, 2, & 3. They give the probability of accepting the lot for various values of $\theta/\theta_\circ$.

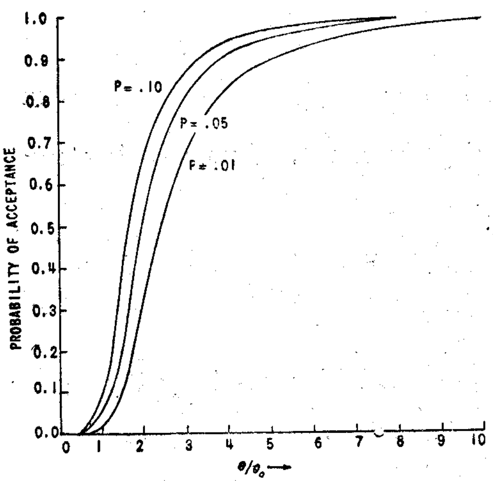

FIG. 1—Operating characteristic curves for Weibull distribution for $r = 5$.

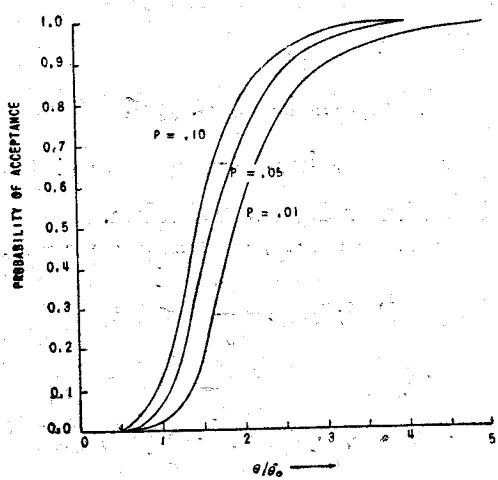

FIG. 2—Operating characteristic curves for Weibull distribution for $r = 10$.

The producer's risk is the probability of rejection of the lot whenever $\theta \geqslant \theta_\circ$ and is given by

$$Pr\left\{\hat{\theta}_{rn} < A/\theta \geqslant \theta_\circ\right\} = Pr\left\{\chi^2_{2r} < \frac{2rA}{\theta_\circ} \cdot \frac{\theta_\circ}{\theta} \Big/ \theta \geqslant \theta_\circ\right\} \quad (11)$$

Thus for any given $\theta \geqslant \theta_\circ$ we can compute the producer's risk for a given sampling plan from χ^2—Table [1]. For a given value of producer's risk, say α, one may be interested to know what value of $\theta/\theta_\circ$ will insure a producer's risk $\leqslant \alpha$, when the sampling plan under discussion is adopted. From equation (11) one may obtain

$$Pr\left\{\chi^2_{2r} \geqslant \left(\frac{2rA}{\theta_\circ} \Big/ \frac{\theta}{\theta_\circ}\right) \Big/ \theta/\theta_\circ \geqslant 1\right\} > 1 - \alpha \quad (12)$$

The minimum value of $(\theta/\theta_\circ)$ satisfying the inequality (12) is given in Table 2, for α = 0·01, 0·05, p^*= 0·01, 0·05, 0·1, and various values of r.

TABLE 2

SHOWING THE MINIMUM θ/θ_0 FOR A GIVEN VALUE OF PRODUCER'S RISK

$r \setminus P^*$	Producer's risk $\alpha=0\cdot05$			Producer's risk $\alpha=0\cdot01$		
	0·01	0·05	0·10	0·01	0·05	0·10
1	89·417	58·165	44·709	458·209	298·060	229·104
2	18·674	13·288	10·941	44·704	31·946	26·192
3	10·283	7·702	6·511	19·280	14·440	12·208
4	7·351	5·674	4·889	12·205	9·421	8·118
5	5·891	4·646	4·058	9·073	7·157	6·250
6	5·017	4·023	3·549	7·342	5·888	5·194
7	4·435	3·604	3·206	6·253	5·083	4·520
8	4·019	3·303	2·957	5·506	4·524	4·051
9	3·707	3·074	2·768	4·962	4·115	3·705
10	3·462	2·895	2·618	4·548	3·803	3·440
11	3·252	2·738	2·487	4·222	3·555	3·163
12	3·104	2·630	2·397	3·959	3·354	3·058
13	2·968	2·528	2·312	3·742	3·188	2·915
14	2·852	2·442	2·240	3·559	3·047	2·795
15	2·752	2·367	2·177	3·403	2·927	2·692

SAMPLE SIZE

It is to be noted that so far in the discussion of the sampling plan and their characteristics n did not occur explicitly anywhere except in the expression for $\hat{\theta}_{rn}$. Thus for a preassigned r, the choice of a sampling plan and its characteristics are in no way affected $\hat{\theta}_{rn}$ by the choice of n, so that one may ask: what is the number of items that are to be put on the test? To answer this question it may be noted that the time to get r failures is a random variable whose distribution depends on n (and also p), so that we may fix n such that the test terminates within a prescribed time t_0 with a probability exceeding a given value P^{**}. Since the distribution of $t_r = y$ is given by the probability density

$$f(y) = \frac{n\,!}{(r-1)\,!\,(n-r)\,!} \exp\left\{-\frac{(n-r+1)y^p}{\theta}\right\} (p/\theta)y^{p-1} \left(1-\exp\left\{\frac{-y^p}{\theta}\right\}\right)^{r-1} \tag{13}$$

the above criterion may be expressed mathematically thus

$$\int_0^{t_0} f(y)\,dy \geqslant P^{**} \tag{14}$$

It can be easily seen that (14) reduces to

$$I_{\exp\{-t^p/\theta\}}(n-r+1, r) \geqslant 1-P^{**} \tag{15}$$

Where $I_x(p,q)$ is the incomplete beta function ratio tabulated by Pearson. Thus for a given r and P^{**} minimum values of n may be obtained from (15) by using Pearson tables[8].

COMPUTATION OF TABLES

Tables 1 and 2 have been computed using equations (9) and (12) respectively with the help of χ^2—Table 1. For computing Table 3 the equation (15) has been used. Since the incomplete beta function ratio $I_x(p, q)$ are available [8] upto the value 30 of the argument, it has been necessary to use the following equivalent or approximate formulae for higher values:

$$I_x\,(n-r+1,\,r) = \sum_{m=0}^{r-1} \binom{n}{m} (1-x)^m\, x^{n-m} \leqslant 1 - P^{**} \qquad (16)$$

and

$$I_x\,(n-r+1,\,r) \simeq P\,\{\chi^2_{2r} > 2n(1-x)\} \leqslant 1 - P^{**} \qquad (17)$$

where $x = \exp\left\{\dfrac{-t^p}{\theta}\right\}$

Table 3 has been prepared for $P^{**}=0\cdot95$ from equations (16) and (17), whenever necessary using Tables at references 9 and 10 respectively. It may be pointed out that the tabulated values of n are always rounded off to nearest higher integer.

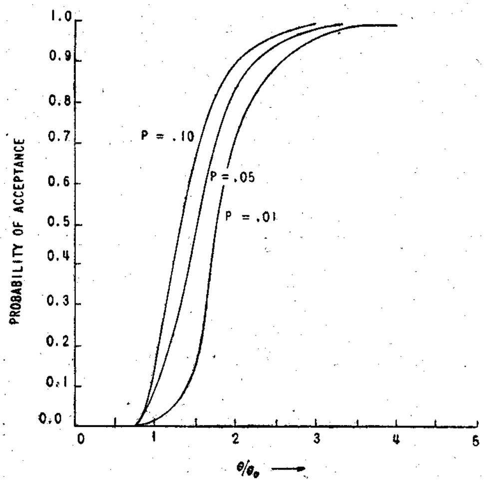

FIG. 3—Operating characteristic curves for Weibull distribution for $r = 15$.

ILLUSTRATIVE EXAMPLES

Example 1: Choice of sampling plan—Assuming that the life distribution follows Weibull distribution with shape parameter p, we went to have sampling plan with $P'=0\cdot5$ and $r=10$. From Table I, we get $\dfrac{A}{\theta_o} = 1\cdot57$. Suppose we take $\theta_o = 1000$ hours, then our sampling plan is such that accept the lot if $\hat{\theta}_{rn} \geqslant 1570$, reject otherwise.

Example 2: Producer's risk—For the plan considered in Example I, suppose one is interested to know what will the quality of the producer's product so that his risk is less than 0·05, the minimum value $\dfrac{\theta}{\theta_o} = 2\cdot895$. Thus the manufacturer's product should have the true mean life at least equal to 2895 hours.

Example 3: Choice of sample size—Given a sampling plan (r, A, θ_o) how many items are to be put under test when the true mean life $\theta = \theta_o$, so that a decision is reached within a given period t_0 with a probability exceeding $P^{**}=0\cdot95$. For the plans considered above, if we want a decision within 500 hours (for the case $p=1$), we see from the Table 3, the minimum sample size is $n=37$. On the other hand if we wish to have a decision within 300 hours we see from Table 3 that the number of items to be put under test is 58.

TABLE 3

THE MINIMUM VALUES OF n FOR VARIOUS VALUES OF $r=1$ (1) 15, $P^{**}=\cdot 95$ AND $x=\cdot 1$ ($\cdot 1$), 6($\cdot 05$), 7($\cdot 02$), $\cdot 9$($\cdot 01$) $\cdot 95$.

r \ x	·1	·2	·3	·4	·5	·6	·65	·7	·72	·74	·76	·78	·80	·82	·84	·86	·88	·90	·91	·92	·93	·94	·95
1	2	2	3	4	5	6	7	9	10	10	11	13	14	16	18	20	24	29	32	36	42	49	60
2	3	4	5	6	8	10	12	14	15	17	18	20	22	25	28	32	38	46	51	58	66	78	94
3	5	6	7	8	11	14	16	19	21	22	24	27	30	33	38	43	51	61	68	78	88	105	126
4	6	7	9	11	13	17	20	24	26	28	30	33	37	41	47	53	64	76	84	95	111	130	156
5	7	9	10	13	16	21	24	28	30	33	36	39	44	49	55	64	74	89	100	115	131	153	184
6	8	10	12	15	18	24	28	33	35	38	41	45	50	56	64	74	85	106	117	132	151	176	211
7	10	12	14	17	21	27	31	37	40	43	47	51	58	64	72	82	96	119	132	149	170	198	237
8	11	13	16	19	23	30	35	41	44	48	52	57	64	70	80	92	110	132	147	165	188	220	260
9	12	14	17	21	26	33	38	45	49	53	57	64	70	78	88	100	121	145	161	181	207	241	289
10	13	16	19	23	28	36	42	49	53	57	62	68	76	84	96	113	131	158	175	197	225	262	316
11	14	17	20	26	30	39	45	53	57	62	68	74	82	92	107	122	142	170	189	213	243	283	340
12	16	19	22	27	33	42	49	57	62	68	74	80	88	98	114	131	152	183	203	228	261	304	365
13	17	20	24	28	35	45	52	61	66	72	78	85	94	109	122	139	163	195	217	244	278	325	389
14	18	21	25	30	37	48	55	65	70	76	84	90	100	115	130	148	173	207	230	259	296	345	414
15	19	23	27	32	40	51	59	70	74	80	88	96	109	122	137	157	183	219	244	274	313	365	438

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REFERENCES

1. GOODE, A. P. & KAO, H. K. *Proc. Seveuth nat. Symp.* on *'Reliability and Quality Control in Electrics'*, Philadelphia, Jaunary 1961 p. 24.
2. GUPTA, S. S., *Technometrics*, 4, 151 (1962).
3. GUPTA, S. S. & GROLL, P. A., *J. Amer. Statist Ass.*, 56, 942 (1961).
4. SOBEL, M. & TISCKENDORP, J. A. *Proc. fifth nat. Symp. 'Reliability and Quality Control,'* Philadelphia. Pa., (1959), 180.
5. KAO, H. K. "Some Statistical Aspects of Life testing of electron tubes", (M. S. Thesis, Department of Industrial Engineering. Coulmbia University, New York).
6. WEIBULL, W., *J. appl. Mech.* 18, (1951).
7. SOBEL, M., *J. Amer. Statist. Ass.*, 48, 486 (1953).
8. PEARSON, K., "Tables of the incomplete Bets function", (Cambrige University Press, Cambridges, England), 1956.
9. Staff of the Harvard Univ. Computation Laboratory. "Table of Cumulative Binomial Probability Distribution" (Harvard University Press), 1955.
10. FISHER, R. A. & YATES, F., "Statistical Tables for Biological, Agricultural and Medical Reseach" fifth edition, (Olliver and Boyd, London), 1957.