HEAT FLOW IN A CYLINDER IN CONTACT WITH WELL STIRRED FLUID

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The transient temperature distribution in a finite hollow cylinder in contact with a well stirred fluid at its outer surface has been obtained. At the inner cylindrical surface cylinder has been assumed to absorb flux which is sinusoidal along the length of the cylinder and the plane ends of the cylinder have been assumed to be impervious to heat. Some numerical results have been exhibited graphically.

Some cases of heat flow in solids in contact with well stirred fluid have been discussed in earlier investigations by March & Weaver[1], Schumann[2], Lowan[3], Jaeger[4], Carslaw & Jaeger[5,6] and Chao & Weiner[7]. In the present paper we consider the transient state combined radial and axial heat flow problem in a finite hollow cylinder in contact with a given mass of well stirred fluid at its outer surface and absorbing heat flux at the inner surface which is sinusoidal along the length of the cylinder. Such a boundary condition is suggested in the case of a nuclear reactor designed for power production with a cylindrical or a rectangular type of reactor core. For such a core the neutron flux density distribution is sinusoidal along the length of the core (Glasstone[8]), and the heat liberation can be assumed to be proportional to the neutron flux density. The plane ends of the cylinder and the core have been assumed to be thermally insulated and the liquid at the outer surface to be so well stirred that its temperature is maintained equal to that of the surface of the cylinder which is thus a function of time only. Initially the entire system is assumed to be at zero temperature and the fluid is assumed to lose heat to the atmoshpere at a rate proportional to its temperature. The problem has been solved by the use of Laplace and finite cosine transforms and a numerical example has been worked out. The results have been exhibited graphically.

STATEMENT OF THE PROBLEM

Consider the heat flow in a cylinder $a < r < b$, $o < z < l$ in contact with mass M of a well stirred fluid at $r=b$. If $\theta\ (r, z, t)$ be the temperature function in the cylinder and $\varphi(t)$ the temperature of the fluid, $\theta\ (r, z, t)$ satisfies the equation

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{k}\frac{\partial\theta}{\partial t} \qquad \begin{matrix} a < r < b \\ o < z < l \end{matrix} \qquad t > o \tag{1}$$

where k is the diffusivity of the material of the cylinder.

$$\theta = \varphi = o, \qquad t = o \tag{2}$$

Boundary conditions

$$-K\frac{\partial\theta}{\partial r} = Q_o \sin\frac{\pi z}{l}, \quad r = a \qquad t > o \tag{3}$$

where Q_o is the flux at the centre of the finite nuclear source in contact with $r=a$ and

$$\left.\begin{aligned} \theta &= \varphi \qquad && r = b \\ \frac{\partial \theta}{\partial z} &= o && z = o \\ \frac{\partial \theta}{\partial z} &= o && z = l \end{aligned}\right\} \quad t > o \tag{4, 5, 6}$$

$\varphi\,(t)$ satisfies the differential equation

$$M\,c'\,\frac{d\,\varphi}{d\,t} + H\,\varphi = -\,2\,\pi\,b\,K \int_0^l \left.\frac{\partial\,\theta}{\partial\,r}\right|_{r=b} dz, \quad t > o \tag{7}$$

where $K = k\,\rho\,c$ is the conductivity of the cylinder, c' the specific heat of the fluid and H is the rate of loss of heat per degree from the fluid to the surroundings.

SOLUTION

Defining $\overline{\theta}\,(r, z, p)$ as the Laplace transform of $\theta\,(r, z, t)$ as

$$\overline{\theta}\,(r, z, p) = \int_0^\infty \theta\,(r, z, t)\; e^{-pt}\,dt \tag{8}$$

The Laplace transform of equation (1) taking into account the initial condition (2) is given by

$$\frac{\partial^2 \overline{\theta}}{\partial r^2} + \frac{1}{r}\,\frac{\partial\,\overline{\theta}}{\partial\,r} + \frac{\partial^2 \overline{\theta}}{\partial z^2} = \left(p/k\right)\overline{\theta} \tag{9}$$

with boundary conditions

$$\frac{\partial\,\overline{\theta}}{\partial\,r} = -\,\frac{Q_0}{pK}\,\sin\,\pi z/l \qquad r = a \tag{10}$$

$$\overline{\theta} = \overline{\varphi} \qquad r = b \tag{11}$$

$$\frac{\partial\,\overline{\theta}}{\partial\,z} = 0 \qquad z = o \tag{12}$$

$$\frac{\partial\,\overline{\theta}}{\partial\,z} = 0 \qquad z = l \tag{13}$$

and equation (7) gives

$$M\,c'\,p\,\overline{\varphi} + H\,\overline{\varphi} + 2\,\pi\,b\,K \int_o^l \left.\frac{\partial\,\overline{\theta}}{\partial\,r}\right|_{r=b} d\,z = 0 \tag{14}$$

If $\overline{\theta}_{cn}(r, p)$ is defined as the finite cosine transform of $\overline{\theta}(r, z, p)$ with respect to z, we have

$$\overline{\theta}_{cn}(r, p) = \int_0^l \overline{\theta}(r, z, p) \cos \frac{n \pi z}{l} \, dz, \qquad n = 0, 1, 2, 3\ 0 \ldots \tag{15}$$

Applying this transform to equation (9) taking into consideration (12) and (13) we get

$$\frac{d^2 \overline{\theta}_{cn}}{dr^2} + \frac{1}{r} \frac{d \overline{\theta}_{cn}}{dr} - \left(p/k + \frac{n^2 \pi^2}{l^2} \right) \overline{\theta}_{cn} = o \tag{16}$$

with conditions

$$\frac{d \overline{\theta}_{cn}}{dr} = 0, \qquad n = 1, 3, 5,$$

$$= - \frac{Q_0}{pK} \frac{l}{\pi (n^2 - 1)} \left\{ (-1)^{n+1} - 1 \right\}, \qquad n = 0, 2, 4, 6, \ldots \tag{17}$$

and

$$\overline{\theta}_{cn}(r, p) = \int_0^l \overline{\Phi} \cos \frac{n \pi z}{l} \, dz = \overline{\varphi} \, l \quad n = o$$
$$= 0, \quad n = 1, 2, 3 \tag{18}$$

The solution of differential equation (16) is

$$\overline{\theta}_{cn}(r, p) = A_n I_o (\mu_n r) + B_n K_o (\mu_n r) \tag{19}$$

where

$$\mu^2_n = p/k + n^2\pi^2/l^2 \tag{20}$$

and I_o and K_o are the modified Bessel functions of order zero.

Determining the constants A_n, B_n from the boundary conditions (17) & (18), we obtain

$$\overline{\theta}_{co}(r, p) = \frac{\overline{\varphi} \, l \, G_{o,1}(\mu_o r, \mu_o a) - \frac{2 \, l \, Q_0}{p \pi K} G_{o,o}(\mu_o r, \mu_o b)}{G_{o,1}(\mu_o b, \mu_o a)} \tag{21}$$

and

$$\overline{\theta}_{cn}(r, p) = - \frac{Q_o \, l \left\{ (-1)^{n+1} - 1 \right\} G_{o,o}\left(\mu \ r, \mu_n b\right)}{\pi \, p \, (n^2 - 1) \, \mu_n \, G_{o,1}(\mu_n b, \mu_n a)}, \qquad n = 2, 4, 6 \tag{22}$$

where

$$G_{m,n}(x, y) = I_m(x) K_n(y) + (-1)^{m+n-1} K_m(x) I_n(y) \tag{23}$$

and therefore

$$\overline{\theta}\,(r, z, p) = \frac{\overline{\theta_{co}}}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \overline{\theta}_{cn} \cos\frac{n\pi z}{l}$$

$$= \frac{\overline{\varphi}\; G_{o,1}(\mu_o\, r, \mu_o\, a) - \dfrac{2\,Q_o}{\mu_o\, p\pi\, K}\; G_{o,o}(\mu_o\, r, \mu_o\, b)}{G_{o,1}(\mu_o\, b, \mu_o\, a)}$$

$$+ \frac{4\,Q_o}{\pi\, p\, K}\sum_{n}\frac{G_{o,o}(\mu_n\, r, \mu_n\, b)\;\cos n\,\pi\, z/l}{\mu_n\,(n^2-1)G_{o,1}\,(\mu_n\, b, \mu_n\, a)} \tag{24}$$

Substituting the value of $\overline{\theta}$ given in (24) in equation (14)

$$\overline{\varphi} = \frac{4\,l\,Q_o}{p\;\mu_o\triangle_p} \tag{25}$$

where

$$\triangle_p = [M\,c'\,p + H\,]\;G_{o,1}\,(\mu_o b, \mu_o\, a) + 2\,\pi\, b\, k\, l\,\mu_o\; G_{1,1}\,(\mu_o b, \mu_o\, a) \tag{26}$$

and substituting (25) in (24)

$$\overline{\theta}\;(r, z, p) = \frac{4\,l\,Q_o\,K\,\pi\,G_{o,1}\,(\mu_o\, r, \mu_o\, a) - 2\,Q_o\triangle_p\,G_{o,0}\,(\mu_o\, r, \mu_o\, b)}{p\,\mu_o\triangle_p\;\; G_{o,1}\,(\mu_o\, b, \mu_o\, a)}$$

$$+\sum_{n}\frac{4\,Q_o\,G_{o,o}\;(\mu_n\, r, \mu_n\, b)\;\cos n\,\pi\, z/l}{K\,\pi\, p\,(n^2-1)\,\mu_n\,G_{o,1}\,(\mu_n\, b, \mu_n\, a)} \tag{27}$$

We now apply the inversion theorem for the Laplace transform to obtain $\theta\,(r, z, t)$ and $\varphi\,(t)$ from $\overline{\theta}\,(r, z, p)$ and $\overline{\varphi}\,(p)$. Accordingly

$$\theta\,(r, z, t) = \frac{1}{2\,\pi\, i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{\theta}\; e^{pt}\,dp\;,\quad \varphi\,(t) = \frac{1}{2\,\pi\, i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{\varphi}\,(p)\; e^{pt}\,dp \tag{28}$$

The integrand on the right hand side of (25) is a single valued function of p with simple poles at $p = o$ and at $p = \alpha_s$ where α_s are the roots of

$$\triangle_p = 0 \tag{29}$$

Finding the residues at these poles and simplifying we get

$$\varphi\;(t) = \frac{4\,Q_o\,l\,a}{H} + 4\,Q_o\,l\sum_{s}\left.\frac{e^{\alpha_s t}}{\alpha_s\;\;\mu_o s\,\triangle'}\right|_{p=\alpha_s} \tag{30}$$

and similarly

$$\theta(r, z, t) = \frac{4\,Q_\circ\, l\, a}{H} - \frac{2\,Q_\circ\, a \log r/b.}{K\,\pi}$$

$$+ \frac{4\,Q_\circ}{K\pi} \sum_n \frac{G_{o,o}(\beta_n r, \beta_n b) \cos n\,\pi\, z/l}{(n^2-1)\,\beta_n\, G_{1,o}(\beta_n a, \beta_n b)}$$

$$+ 4\,Q_\circ\, l \sum_s \frac{G_{o,1}(\mu_{o,s} r, \mu_{o,s} a)\, e^{\alpha_s t}}{\triangle'_{p=\alpha_s}\, \alpha_s.\, \mu_s.\, G_{o,1}(\mu_{o,s} b, \mu_{o,s} a)}$$

$$+ \frac{2\,Q_\circ}{K\pi} \sum_J \frac{e^{\lambda_{oj} t}\,[G_{o,1}(\mu_{,j}\, r, \mu_{o,j} a) - b\, \mu_{o,j}\, G_{1,1}(\mu_{o,j} b, \mu_{o,j} a)\, G_{o,o}\, \mu_{oj} r, \mu_{oj} b)]}{\mu_{o,j}\, D'\lambda_{o,j}}$$

$$+ \frac{4\,Q_\bullet}{K\,\pi} \sum_n \sum_J \frac{G_{o,o}(\mu_{n,j} r, \mu_{n,j} b)}{(n^2-1)\,\mu_{n,j}\, D'\lambda_{n_J}} \frac{e^{\lambda_{nj} t} \cos n\,\pi\, z/l}{} ,n = 2, 4, 6 \qquad (31)$$

where λ_{nj} are the roots of the equation

$$D_p = G_{o,1}(\mu_n b, \mu_n a) = 0 \qquad (32)$$

$$\beta_n = n\,\pi/l. \qquad (33)$$

$$\triangle'_{p=\alpha_s} = \left| \frac{d}{dp}\, \triangle_p \right|_{p=\alpha_s} \qquad (34)$$

$$\mu^2_{nj} = \frac{\lambda_{nj}}{k} + \frac{n^2\,\pi^2}{l^2}, \qquad \mu^2_{o,s} = \alpha_s/k \qquad (35)$$

$$D'\lambda_{nj} = \left| \frac{d}{dp}\, D_p \right|_{p=\lambda_{nj}} \qquad (36)$$

VERIFICATION OF THE SOLUTION

The solution given in equations (30) and (31) satisfy the equation (7) by virtue of equation (26). It is also seen that at $r=b$, $\theta=\phi$, thus satisfying the boundary condition (4) since $G_{o,o}(x, x) = 0$.

At $r = a$ we get from (31)

$$-K \frac{\partial \bar{\theta}}{\partial r} = \frac{2}{\pi} + \sum_n \frac{4}{n^2-1} \cos \frac{n\,\pi\, z}{l}, \qquad n = 2, 4, 6$$

which is the Fourier cosine expansion for $\sin \frac{\pi z}{l}$ thus satisfying the boundary condition (3). To prove that the solution satisfies the initial condition (2) we go back to equations (25 and (28). The contour of integration is a straight line parallel to the imaginary axis at a distance γ from it. Since α_s, the roots of equation (26) are all real and negative we can choose γ as large as we please. By replacing the Bessel functions by their asymptotic expansions for large p and retaining the dominant terms we find that the integral on the right hand side of (25) at $t=0$ is, apart from a constant factor,

$$\backsim \left(\frac{b}{a} \right)^{1/2} \int_{\xi - i\infty}^{\xi + i\infty} \frac{\exp\{-\mu (b-a)\}}{p^2} \, dp$$

where $p = \xi + i\,\eta$. It can be shown to be less than

$$\left(\frac{b}{a} \right)^{1/2} \exp \left\{ - \sqrt{\frac{\xi}{2\,k}} \;(b-a) \right\}$$

which shows that it can be made arbitrarily small by taking large values of ξ and therefore the integral on the right hand side of (28) vanishes at $t=0$ thus satisfying the initial condition (2). It can similarly be shown that $\theta\,(r, z, t)$ also satisfies the initial condition.

NUMERICAL EXAMPLE

For numerical work it is found to be convenient to express the solution (30) and (31) in a slightly different form. For this we make the following substitutions

$$\mu_{os}\, a = i\,\eta_s \qquad \mu_{os}\, b = i\,\rho_1\,\eta_s \qquad \rho_1 = b/a$$

$$\mu_{nj}\, a = i\,\zeta_{nj} \qquad \mu_{nj}\, b = i\,\rho_1\,\zeta_{nj}$$

where

$$\zeta_{nj}^{\;2} = -\,a^2 \left(\frac{\lambda_{\,nJ}}{k} + \frac{n^2\,\pi^2}{l^2} \right)$$

equation (26) can be put in the form

$$\frac{S_{1,\,1}\,(\rho_1\;\eta_s,\,\eta_s\,)}{S_{o,\,1}\,(\rho_1\;\eta_s,\,\eta_s\,)} = \frac{k'\eta_s{}^2 - H'}{\eta_s} \tag{37}$$

$$\text{where } H' = \frac{H}{2\pi bKl}, \qquad k' = \frac{Mc'}{2\pi bcpl},$$

and equation (32) can be put in the form

$$S_{o,1}(\zeta_{nj}\;\rho_1,\,\zeta_{nj}) = o \tag{38}$$

where

$$S_{m,n}\left(x, y \right) = J_m\,(x)\,Y_n\,(y) - Y_m\,(x)\,J_n\,(y) \tag{39}$$

and the solution (30) and (31) as

$$\varphi\,(t) = \frac{4\,Q_o\,l\,a}{H} - 8\,Q_o\,l \sum_s \frac{a^2 e^{-\,(\,k\eta_s^2/a^2\,)\,t}}{\Delta_1'\,\eta^2_s} \tag{40}$$

and

$$\theta(r, z, t) = \frac{4 Q_\circ l a}{H} + \frac{2 Q_\circ a \log r/b}{K \pi} + \frac{4 Q_\circ}{K \pi} \sum_n \frac{G_{o,o} (\beta_n r, \beta_n b) \cos n \pi z/l}{(n^2 - 1) \beta_n \; G_{1,o} (\beta_n a, \beta_n b)}$$

$$- 8 Q_\circ l \sum_s \frac{S_{o,1} (\eta_s r/a, \eta_s) \; e^{-(k\eta^2 s/a^2) t}}{- \, \eta^2_s S_{o,1} (\eta_s \rho_1, \eta_s) \; \triangle_1}$$

$$+ \frac{4Q_\circ}{K\pi} \sum_J \frac{e^{-k\xi_{o,s}^2 t/a^2} \left\{ a^2 S_{0,1} (\zeta_{o,j} r/a, {}_o\zeta_j) - \pi/_2, b \zeta_{oj} S_{1,1} (\zeta_{oj} \rho_1, \zeta_{oj}) S_{0,1} (\mu_{oj} r/a, \mu_{oj} \rho_1) \right\}}{D'_1 (\xi) \xi_{oJ}^2}$$

$$+ \frac{8Q_\circ}{K\pi} \sum_n \sum_J a^2 \; e^{-k \left(\frac{\xi^2_{nJ}}{a^2} + \frac{n^2 \pi^2}{l^2} \right) t} \frac{S_{0,0} (\zeta_{nJ} r/a, \zeta_{nj} \rho_1) \cos n \pi z/l}{D'_{1n} (\zeta_{nj}^2) + \gamma_n^2)} \quad (41)$$

where $n = 2, 4, 6,$

$$- D'_1 (\zeta_{nj}) = b S_{1,1} (\zeta_{nj}) \rho, \zeta_{nJ}) + a S_{0,0} (\zeta_{nJ} \rho_1, \zeta_{nJ}) \quad (42)$$

and

$$a^2 \triangle'_1 = \frac{S_{o,1} (\eta_s \rho_1, \eta_s)}{2 \pi l \eta_s b} \left[a^2 H - M c' k \eta^2_s - 4 \pi^2 b^2 k^2 l^2 \eta^2_s \right] + \left[(Ha^2 - Mc'k\eta^2_s) S_{o,o} (\eta_s \rho_1, \eta_s) + 2\pi bkl\eta_s S_{o,1} (\eta_s \rho_1, \eta_s) \right] \quad (43)$$

The values of various parameters chosen for the numerical example are given as follows :—

$a = 1$ $\qquad b = 1 \cdot 5$ $\qquad l = 10$

$k = 0 \cdot 12$ $\qquad \rho = 7 \cdot 85$ $\qquad C = 0 \cdot 118$

and choosing H' and k' each equal to unity we have for the example discussed here $M = 32 \cdot 387\pi$, $H = 3 \cdot 886\pi$. Equation (37) was solved graphically for these values of parameters by plotting both sides of the equation as function of η_s. The points of intersection yield η_s. The first three values of η_s for which equation (37) is satisfied are found to be $\eta_1 = 2 \cdot 659$, $\eta_2 = 9 \cdot 322$, $\eta_3 = 15 \cdot 635$
The values of ζ_{n_1}, ζ_{n_2} and ζ_{n_3}, the first three roots of the equation (38) were also tained graphically and the values of λ_{nJ} calculated. These are tabulated in Table 1.

TABLE 1

n \ J =	1	2	3
=0	1·3905	10·8848	29·8627
2	1·4378	10·9322	29·9694
4	1·5900	11·0743	30·0516
6	1·8168	11·3112	30·2884

Values of $\theta\ (r, z, t)$ were calculated for various values of r and t at $z = l/_2$ and also $\varphi(t)$.

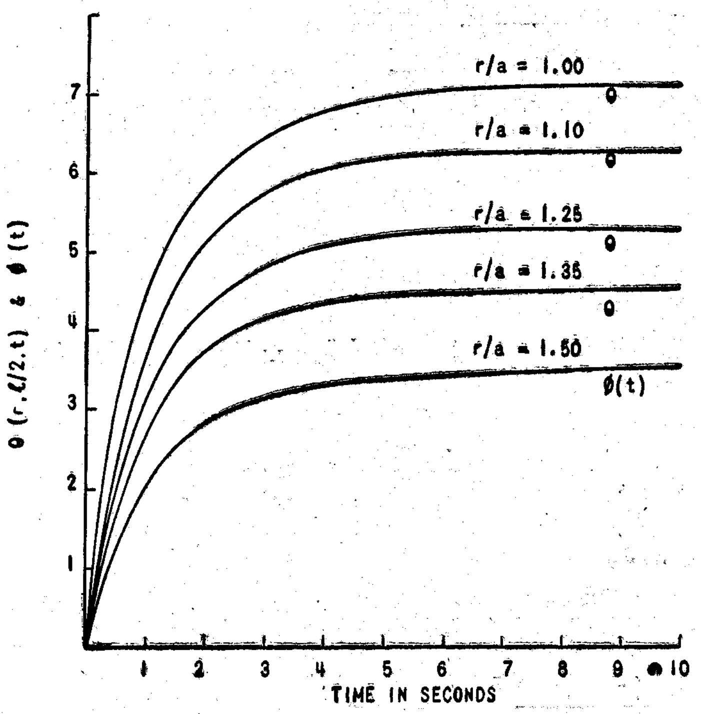

FIG. 1—Temperature distribution in a finite hollow cylinder in contact with well stirred fluid.

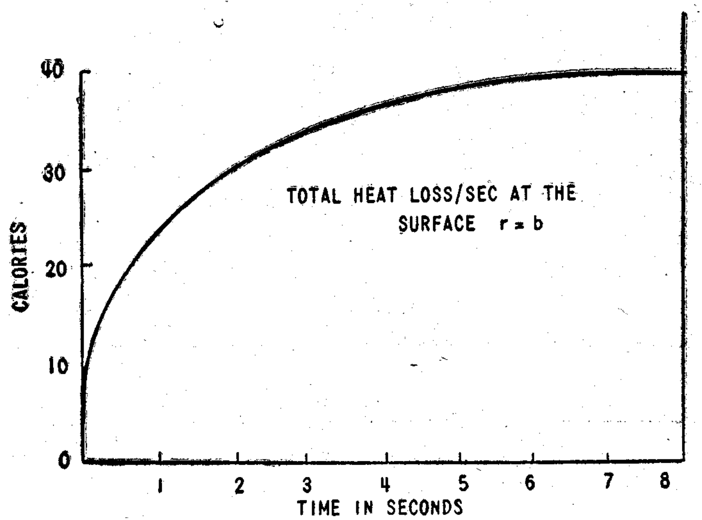

FIG. 2—Total heat loss from the outer surface of the cylinder.

To find the total heat loss at $r=b$, we find the value of the expression $-\,2\pi bk \int_0^l \frac{\partial \theta}{\partial r} \Big|_{r=b} dz$

The results are exhibited in figures (1) and (2). The steady state temperatures in the cylinder at various distances from the cylinder axis are exhibited in figure (3).

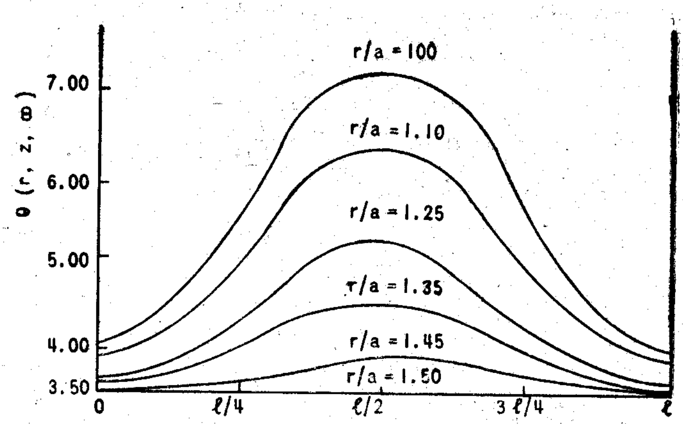

FIG. 3—Steady state temperature in a finite hollow cylinder absorbing sinusoidal flux at the inner surface and in contact with well stirred fluid at the outer surface.

Equation (37) has also been solved for various values of H' and k' and the first three values of η_s are given in Table 2.

TABLE 2

	k′	η_1	η_2	η_3
0·5	0·5	9·114	15·528	21·820
0·5	1·0	2·730	9·323	15·648
1·0	1·5	2·984	9·390	15·685
2·0	1·0	9·319	15·646	21·905
2·0	1·5	2·938	9·388	15·688
2·0	2·0	3·088	9·421	15·706

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REFERENCES

1. MARCH, H. W. & WEAVER, W., *Phys. Rev.*, **31**, 1072 (1928).
2. SCHUMANN, T. E. W., *Phys. Rev.*, **37**, 1508 (1931).
3. LOWAN, A. N., *Phil. Mag.*, **17**, 849 (1934).
4. JAEGER, J. C., *Proc. Roy. Soc.*, **74**, 342 (1940) & **75**, 130 (1941).
5. CARSLAW, H. S. & JAEGER, J. C., "Operational Methods in Applied Mathematics". (Oxford), 1948, p. 287.
6. CARSLAW, H. S. & JAEGER, J. C., "Heat Conduction in Solids", (Oxford), 1959., p. 138 & 329,
7. CHAO, C. C. & WEINER, J. H., *Quart. Appl. Math.*, **14**, 214 (1956).
8. GLASSTONE, S., "Principles of Nuclear Reactor Engineering", (D. Van Nonstrand, New York), 1955, p. 67.