MANY-SERVER QUEUING PROBLEM WITH LIMITED WAITING SPACE

K. L. Arora

Weapons Evaluation Group*, Delhi

ABSTRACT

In this paper the multi-channel queuing system with finite waiting space has been considered under Poisson arrival and exponential service time distributions. The Laplace transform of the time-dependent probability generating function has been obtained. In two and three channel cases probabilities have been derived explicitly and known equilibrium results are shown to hold. At the end are given the tables for mean number of units in the system, with two and three channels, and the probability that an arriving unit on finding the system occupied is lost.

INTRODUCTION

Much stress has been observed to be laid on finite waiting space problems in queuing theory for its utility to the industry and to other spheres in practical life. The steady-state solution of the multi-server queuing system with finite waiting space has been studied by Jackson[3]. The solution of the single server system has been given by Heathcote and Moyal[1]. It was however Saaty[2] who studied the transient behaviour of the multiple-channels queuing problem with infinite waiting space.

In this paper an attempt is made to solve many-server queuing problem with finite waiting space. The Laplace transform of the time-dependent generating function has been obtained. In the case of two and three channels, the probabilities have been expressed explicitly in Laplace transforms, the inversions of which involve prohibitive Mathematics. The steady-state solutions have been derived by using the corollary of Abel's Theorem.

THE PROBLEM AND ITS SOLUTIONS

Assume c service channels. Let the maximum number of units which can be accommodated in the channels and the waiting space be N. Suppose the system (units in waiting queue and under service) starts with i ($> c - 2$) units. Define P_n (t) as the probability that at time t there are n units in the system, so that $P_n(0) = \delta_{in}$, the Kronecker delta. The forward difference-differential equations can easily be seen to be—

$$dP_o(t)/dt = -\lambda P_o(t) + \mu P_1(t) \tag{1}$$

$$dP_n(t)/dt = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad 1 \leqslant n < c \tag{2}$$

$$dP_n(t)/dt = -(\lambda + c\mu) P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t), \quad c \leqslant n < N \tag{3}$$

$$dP_N(t)/dt = -c\mu P_N(t) + \lambda P_{N-1}(t) \tag{4}$$

*Now Scientific Evaluation Group, Delhi.

Introducing the generating function

$$F(x,t) = \sum_{n=0}^{N} P_n(t)x^n, \text{ so that } F(x,0) = x^i \tag{5}$$

Multiplying (1) through (4) with appropriate powers of x and adding, we get

$$dF(x,t)/dt = -\left(\lambda + c\mu - \lambda x - \frac{c\mu}{x}\right) F(x,t) + \mu \left(1 - \frac{1}{x}\right) \sum_{n=0}^{c-1}(c - n)P_n(t)x^n + \lambda(1 - x)\, x^N P_n(t) \tag{6}$$

Applying Laplace transform viz., $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$ to (6), we have

$$\bar{F}(x,s) = \frac{x^{i+1} + \lambda x^{N+1}(1-x)\bar{P}_N(s) - \mu(1-x)\sum_{n=0}^{c-1}(c - n)\bar{P}_n(s)x^n}{-[\lambda x^2 - (s + \lambda + c\mu)x + c\mu]} \tag{7}$$

The denominator on right of (7) has two zeros in

$$\alpha_{1,2}(s) = [s + \lambda + c\mu \pm ((s + \lambda + c\mu)^2 - 4c\lambda\mu]/2\lambda)^{\frac{1}{2}}$$

and since $\bar{F}(x, s)$ is regular for all domains of x, the numerator must vanish at $x = \alpha_1$ (for +ve root) and $x = \alpha_2$ (for —ve root) giving rise to two relations

$$\alpha_1^{i+1} + \lambda\alpha_1^{N+1}(1 - \alpha_1)\bar{P}_N(s) - \mu(1 - \alpha_1)\sum_{n=0}^{c-1}(c - n)\bar{P}_n(s)\alpha_1^n = 0 \tag{8}$$

$$\alpha_2^{i+1} + \lambda\alpha_2^{N+1}(1 - \alpha_2)\bar{P}_N(s) - \mu(1 - \alpha_2)\sum_{n=0}^{c-1}(c - n)\bar{P}_n(s)\alpha_2^n = 0 \tag{9}$$

involving $(c + 1)$ unknowns.

The other $(c - 1)$ relations are

$$\bar{P}_n(s) = \frac{\sqrt{(s/\mu)}}{n!\sqrt{(s/\mu + 1)}}\Bigg\{(c - 1)\bar{P}_{c-1}(s)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\frac{\lambda}{\mu}} \frac{d^k}{dz^k}\phi_1\left[\frac{s}{\mu}, -(c-2), 1, (1-z), \frac{\lambda}{\mu}(1-z)\right]_{z=0}$$

$$- \frac{\lambda}{\mu}\bar{P}_{c-2}(s)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\lambda}{\mu}\right)^{n-k} e^{-\frac{\lambda}{\mu}}$$

$$\frac{d^k}{dz^k}\phi_1\left[\frac{s}{\mu}, -(c-1), 1, (1-z), \frac{\lambda}{\mu}(1-z)\right]_{z=0}\Bigg\}, \; 0 \leqslant n \leqslant c-2 \tag{10}$$

as given by Saaty[2], equation (21).

Solving equations (8), (9) and (10) and substituting the values of $\overline{P}_n(s)$, $(o \leqslant n \leqslant c-1)$ and $\overline{P}_N(s)$ in (7), $\overline{F}(x, s)$ can be known completely, from which different queue parameters can be determined.

Particular Cases :

I. $c=2$

For $c=2$, the Laplace transform of the generating function (7) is given by

$$\overline{F}(x,s) = \frac{x^{i+1} + \lambda x^{N+1}(1-x)\overline{P}_N(s) - \mu(1-x)[2\overline{P}_0(s) + x\overline{P}_1(s)]}{-[\lambda x^2 - (s+\lambda+2\mu)x + 2\mu]} \tag{11}$$

and the equations involving the unknowns $\overline{P}_0(s)$, $\overline{P}_1(s)$ and $\overline{P}_N(s)$ are

$$\begin{aligned} \beta_1^{i+1} + \lambda\beta_1^{N+1}(1-\beta_1)\overline{P}_N(s) - \mu(1-\beta_1)[2\overline{P}_0(s) + \beta_1\overline{P}_1(s)] &= 0, \\ \beta_2^{i+1} + \lambda\beta_2^{N+1}(1-\beta_2)\overline{P}_N(s) - \mu(1-\beta_2)[2\overline{P}_0(s) + \beta_2\overline{P}_1(s)] &= 0 \end{aligned} \tag{12}$$

and $\quad (s+\lambda)\,\overline{P}_0(s) = \mu\,\overline{P}_1(s)$

where $\quad \beta_1 = \alpha_1/c_{\,=2}, \quad \beta_2 = \alpha_2/c_{\,=2}$

Solving equations (12), we get

$$\overline{P}_1(s) = [(s+\lambda)/\mu]\,\overline{P}_0(s),$$

$$\overline{P}_N(s) = \{[2\mu + (s+\lambda)\beta_1]/\lambda\beta_1^{N+1}\}\overline{P}_0(s) - [\lambda(1-\beta_1)\,\beta_1^{N-i}]^{-1}$$

$$\overline{P}_0(s) = \frac{\left(\frac{2\mu}{\lambda}\right)^{i+1}\left[(1-\beta_2)\beta_2^{N-i} - (1-\beta_1)\beta_1^{N-i}\right]}{(1-\beta_1)(1-\beta_2)[\{2\mu+(s+\lambda)\beta_1\}\beta_2^{N+1} - \{2\mu+(s+\lambda)\beta_2\}\beta_1^{N+1}]} \tag{13}$$

Steady State

We make use of the well-known property of Laplace transform i.e. $\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$, if the limit on the right exists, (11) and (13) under steady state give

$$F(x) = \frac{1 + \rho x - 2\rho^{N+1}x^{N+1}}{1-\rho x}\,.P_o \tag{14}$$

$$P_o = \frac{1-\rho}{1+\rho-2\rho^{N+1}}\text{ , and} \tag{15}$$

$$P_n = 2\rho^n P_o \;,\quad 1 \leqslant n \leqslant N \tag{16}$$

where $\underset{s\to 0}{\text{Limit}}\ \beta_1 = \frac{1}{\rho}$, $\underset{s\to 0}{\text{Limit}}\ \beta_2 = 1$, ρ being the utilising factor. Denoting by L, the mean number of units in the system, we have

$$L = \left.\frac{dF(x)}{dx}\right|_{x=1} = \frac{2\rho[1-\rho^N\{1+N(1-\rho)\}]}{(1-\rho)(1+\rho-2\rho^{N+1})} \tag{17}$$

Letting $N \to \infty$

$$L = \frac{2\rho}{1-\rho^2}\text{, a known result.}$$

II. $c=3$

For $c=3$, the Laplace transform of the generating function (7) reduces to

$$\bar{F}(x, s) = \frac{x^{i+1} + \lambda x^{N+1}(1-x)\bar{P}_N(s) - \mu(1-x)\sum_{n=0}^{2}(3-n)\bar{P}_n(s)x^n}{-[\lambda x^2 - (s+\lambda+3\mu)x + 3\mu]} \tag{18}$$

The relations in four unknowns $\bar{P}_0(s)$, $\bar{P}_1(s)$, $\bar{P}_2(s)$ and $\bar{P}_N(s)$ are

$$\gamma_1^{i+1} + \lambda(1-\gamma_1)\gamma_1^{N+1}\bar{P}_N(s) - \mu(1-\gamma_1)[3\bar{P}_\circ(s) + 2\gamma_1\bar{P}_1(s) + \gamma_1^2\bar{P}_2(s)] = 0,$$

$$\gamma_2^{i+1} + \lambda(1-\gamma_2)\gamma_2^{N+1}\bar{P}_N(s) - \mu(1-\gamma_2)[3\bar{P}_\circ(s) + 2\gamma_2\bar{P}_1(s) + \gamma_2^2\bar{P}_2(s)] = 0,$$

$$(s+\lambda)\bar{P}_\circ(s) = \mu\bar{P}_1(s),$$

$$2\mu\bar{P}_2(s) - (s+\lambda+\mu)\bar{P}_1(s) + \lambda\bar{P}_\circ(s) = 0,$$

where $\gamma_1 = \alpha_1/_{c=3}$, $\gamma_2 = \alpha_2/_{c=3}$

Solving the above, we get

$$\left.\begin{aligned}
\bar{P}_\circ(s) &= \frac{\left(\frac{3\mu}{\lambda}\right)^{i+1}\left[(1-\gamma_2)\gamma_2^{N-i} - (1-\gamma_1)\gamma_1^{N-i}\right]}{\mu(1-\gamma_1)(1-\gamma_2)\left[3(\gamma_2^{N+1} - \gamma_1^{N+1}) + 2\gamma_1\gamma_2(\gamma_2^N - \gamma_1^N)\left(\frac{s+\lambda}{\mu}\right) + \gamma_1^2\gamma_2^2(\gamma_2^{N-1} - \gamma_1^{N-1})\left\{\frac{(s+\lambda)^2+s\mu}{2\mu^2}\right\}\right]}\\
\bar{P}_1(s) &= \left(\frac{s+\lambda}{\mu}\right)\bar{P}_\circ(s),\\
\bar{P}_2(s) &= \{[(s+\lambda)^2+s\mu]/2\mu^2\}\bar{P}_\circ(s), \text{ and}\\
\bar{P}_N(s) &= \frac{\mu}{\lambda\gamma_1^{N+1}}\left[3 + 2\gamma_1\left(\frac{s+\lambda}{\mu}\right) + \frac{\gamma_1^2}{2\mu^2}\left\{(s+\lambda)^2 + s\mu\right\}\right]\bar{P}_\circ(s)\\
&\quad - [\lambda(1-\gamma_1)\gamma_1^{N-i}]^{-1}
\end{aligned}\right\} \tag{19}$$

With the help of (18) and (19), $\bar{F}(x,s)$ can be obtained from which different queue parameters can be evaluated.

For the steady state, following the method employed in case I, it can be seen that the relevant probabilities are

$$\left.\begin{aligned} P_o &= \frac{2(1-\rho)}{2+4\rho+3\rho^2-9\rho^{N+1}}, \\ P_1 &= 3\rho P_o, \\ P_n &= \frac{(3)^3}{3!}\rho^n P_o, \; 2 \leqslant n \leqslant N \end{aligned}\right\} \tag{20}$$

Also the generating function (18) reduces to

$$F(x) = \frac{2+4\rho x+3\rho^2x^2-9\rho^{N+1}x^{N+1}}{2(1-\rho x)} \cdot P_o \tag{21}$$

Mean number of units in the system for this case is

$$L = \frac{3\rho\{2+2\rho-\rho^2-3\rho^N[1+N(1-\rho)]\}}{(1-\rho)\{2+4\rho+3\rho^2-9\rho^{N+1}\}} \tag{22}$$

Letting $N \to \infty$, it becomes

$$L = \frac{3\rho(2+2\rho-\rho^2)}{(1-\rho)(2+4\rho+3\rho^2)}$$

NUMERICAL RESULTS

With the help of the relations (16) and (20)

$$\begin{aligned} P_N &= \frac{2(1-\rho)\rho^N}{1+\rho-2\rho^{N+1}}, \text{ for } c=2 \\ &= \frac{9(1-\rho)\rho^N}{2+4\rho+3\rho^2-9\rho^{N+1}}, \text{ for } c=3 \end{aligned}$$

where P_N is defined as the probability that an arriving unit is lost on finding the system occupied. The numerical values of P_N have been computed for various sets of ρ and N the results are given in Tables 1 and 2.

TABLE I

NUMERICAL VALUES OF P_N

c=2

N \ ρ	·1	·3	·5	·7	·9
2	·01639	·10112	·20	·28994	·36652
3	·00164	·02944	·09091	·16872	·24804
4	·00016	·00876	·04348	·10563	·18250
5	·00002	·00262	·02128	·06885	·14108

Table 2

Numerical Values of P_N when c=3

N \ ρ	·1	·3	·5	·7	·9
3	·00333	·05007	·13433	·22538	·30874
4	·00033	·01480	·06294	·13627	·21744
5	·00003	·00442	·03051	·08708	·16367
6	·000003	·00132	·01503	·05746	·12839

Again employing the relations (17) and (22), the numerical values of L have been tabulated for various sets of ρ and N in Tables 3 and 4.

Table 3

Numerical Values of L when c=2

N \ ρ	0·1	0·3	0·5	0·7	0·9
2	0·197	0·539	0·800	0·994	1·140
3	0·201	0·612	1·000	1·333	1·602
4	0·202	0·641	1·120	1·614	2·039
5	0·202	0·653	1·213	1·847	2·457

Table 4

Numerical Values of L when c=3

N \ ρ	0·1	0·3	0·5	0·7	0·9
3	0·299	[illegible]	1·299	1·627	1·866
4	0·300	0·901	1·469	1·950	2·330
5	0·300	0·920	1·576	2·216	2·767
6	0·300	0·926	1·643	2·433	3·182

Acknowledgements—The author records his gratitude to Dr. R. S. Varma, Professor of Mathematics, University of Delhi, for his kind supervision and to Col. Pritam Pal Singh, Director, Weapons Evaluation Group, for encouragement and permission to publish the paper.

References

1. Heathcote, C. R. and Moyal, J. E., *Biometrika*, **46, 409 (1959)**.
2. Saaty, T. L., *Opns. Res.*, **8**, 755 (1960).
3. Jackson, J. R., *Nav. Res. Log. Quart.*, **5, 315** (1958).
4. Syski, R., "Introduction to Congestion Theory in Telephone Systems", (Oliver and Boyd), 1960.
5. Morse, P. M., "Queues, Inventories and Maintenance", (Wiley, New York), 1958.