A GENERALISATION OF QUEUING WITH BREAKDOWNS

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ABSTRACT

In this paper, a queuing model has been considered in which the service facility is attending to units from a finite population as well as to the units from another independent infinite population. The units from the finite population has "Preemptive resume priority" discipline over the units from the infinite population. The queue length distribution and the stochastic law of busy periods have been obtained by employing some finite discrete transforms. Various interesting particular cases of this model have also been discussed.

INTRODUCTION

The usual assumption in the studies[1,2,3,4,5] dealing with the priority assignment in waiting line problems, is that the units of various priority classes emanate from independent infinite populations. But there are many practical situations in military and industrial fields (e.g. chemical or textile industry, maintenance centres etc.) in which the units of different categories come only from finite populations. A situation of this type has been considered in this paper and some characteristic measures of model have been evaluated.

There are two classes of units which arrive at the service facility for getting service. The units of one type come from a finite population consisting of N members, while the units of the other type come from an infinite population. The inter-arrival times and the service times of the two types of units are governed by some probabilistic law. There are several types of priority disciplines which we can impose. In this paper we consider only the preemptive resume priority discipline. Under this discipline, the units from the finite population displace the units from the infinite population if they are under service, and the displaced unit resumes the service at the point where it was preempted.

It may be remarked that this model has been considered by Avi-Itzhak and Naor[6] under the assumption of Poisson arrivals and exponential service times. The first two moments of the non-priority queue length were obtained by elementary probability considerations.

The problem considered here can be interpreted in many ways. Two different interpretations, we give below for this model. The arrivals and the service of the priority class units can be considered as an interruption process from the view point of low-priority units. This has been studied by Gaver,[7] Keilson[8] and the author[9] in the particular case of a single interruption. In reference (7) and (8) the model has been treated through completion times where as in reference (9), the same process has been studied by restricting the priority class queue size to unity. The model considered in this paper generalises the problem in reference (9), in the sense that interruption occurs from a finite number of sources and therefore a finite number of interruptions can occur.

The problemcan also be interpreted in another way. Let us consider a server who is attending to afinite [illegible] of units say N machines. He also attends to some other jobs emanating fom an another infinite population when he is free from attending to the machines. This model can be utilised to allot the number of machines to the server either by properly balancing between the operational efficiency and the machine availability defined by Benson and Cox[10] or by striking a balance between the mean number of machines and the units waiting. For this model we can intutively see that the machine availability will be the same as in the case of the simple machine interference problem where the server attends only to machines while the operational efficiency is different and depends upon the traffic intensity of low priority units.

In this paper, the queue length distribution and the stochastic law of busy periods for the model mentioned, have been obtained by incorporating all the necessary supplementary variables to make the process, a Markovian one.

THE ASSUMPTIONS IN THE MODEL

Let us suppose that a service facility is attending to units from a finite population consisting of N units, (we designate them as priority units) as well as units from an infinite population (we designate them as ordinary units). Each of the priority units arrives at the facility at random, with mean rate λ_1 if the server is occupied, and with mean rate λ_1^* if server is free. This type of formulation facilitates to discuss two particular cases of the model viz $\lambda_1 = \lambda_1^*$ and $\lambda_1^* = o$. In the former case, the priority units can arrive at the facility even when there is no ordinary unit in the system whereas in the later case, they can arrive only when there are ordinary units in the system. The service times of the priority units are identically and independently distributed random variables with a common probability distribution having the density $S_1(x)$. After getting service, the priority units return to the original population. Let us also assume that the ordinary units arrive at the facility in a Poisson stream with mean rate λ_2 and their service times are identically and independently distributed random variables with an arbitrary probability density $S_2(x)$. Let us impose the preemptive resume priority discipline. According to this discipline, the priority units, on their arrivals replace the ordinary units if it is under service at the service facility, and the displaced ordinary unit on its reentry, which is possible only after the service of all the priority units, waiting, had finished, resumes the service where it was preempted.

FORMULATION OF THE MODEL

Let us define the following probabilities :—

(1) $P_{m,n}(x, y, t)\,dx\,dy\ [1 \geqslant m \leqslant N\,;\ n \geqslant 1]$, the probability that at time t, there are m priority units and n ordinary units in the system, the priority unit under service has the elapsed service time lying between x and $x + dx$ and the ordinary unit was preempted earlier when its elapsed service time lay between y and $y + dy$.

(2) $Q_{m,n}(x, t)\,dx\ [1 \leqslant m \leqslant N\,;\ n \geqslant o]$ — the probability at time t that the system being in the same state as in (1) except that none of the ordinary units was preempted earlier.

(3) $U_n(y, t)\,dy\ [n \geqslant 1]$ — the probability that at time t, there are n ordinary units in the system and the unit under service has the elapsed service time lying between y and $y + dy$.

(4) $P_o(t)$ — the probability that at time t, the system is empty *i.e.* neither priority nor ordinary unit is in the system.

The above states are mutually exclusive and exhaustive and provide the Markovian characterisation of the problem under consideration. Now it is easy to construct the differential difference equations for the process by connecting the various state probabilities at time t and $t+\triangle$ and letting $\triangle \to o$. Thus we have,

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+[(N-m)\lambda_1+\lambda_2+\eta_1(x)]\right\}P_{m,n}(x,y,t)$$

$$=(N-m+1)\lambda_1 P_{m-1,n}(x,y,t)+\lambda_2 P_{m,n-1}(x,y,t) \quad (1)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+[(N-m)\lambda_1+\lambda_2+\eta_1(x)]\right\}Q_{m,n}(x,t)$$

$$=(N-m+1)\lambda_1 Q_{m-1,n}(x,t)+\lambda_2 Q_{m,n-1}(x,t) \quad (2)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+[N\lambda_1+\lambda_2+\eta_2(y)]\right\}U_n(y,t)$$

$$=\lambda_2 U_{n-1}(y,t)+\int_o^\infty P_{1,n}(x,y,t)\,\eta_1(x)\,dx \quad (3)$$

$$\left\{\frac{d}{dt}+[N\lambda_1^*+\lambda_2]\right\}P_o(t)$$

$$=\int_o^\infty Q_{1,o}(x,t)\,\eta_1(x)\,dx+\int_o^\infty U_1(y,t)\,\eta_2(y)\,dy \quad (4)$$

where $\eta_1(x)\triangle$ and $\eta_2(x)\triangle$ are the first order probabilities that the priority and ordinary units respectively complete the service between x and $x+\triangle$ subject to the condition that the units have not completed their service upto time x.

The relation governing $S_i(x)$ and $\eta_i(x)$ is given by

$$S_i(x)=\eta_i(x)\exp\left\{-\int_o^x \eta_i(x)\,dx\right\} \quad (i=1,2) \quad (5)$$

The above equations are to be solved subject to the following boundary conditions:

$$P_{N,n}(o,y,t)=o \quad \text{for all } n \quad (6)$$

$$P_{m,n}(o,y,t)=\int_o^\infty P_{m+1,n}(x,y,t)\,\eta_1(x)\,dx \quad [1<m<N] \quad (7)$$

$$P_{1,n}(o,y,t)=\int_o^\infty P_{2,n}(x,y,t)\,\eta_1(x)\,dx+N\lambda_1 U_n(y,t) \quad (8)$$

$$Q_{N,n}(o,t)=o \quad (9)$$

$$Q_{m,n}(o,t)=\int_o^\infty Q_{m+1,n}(x,t)\,\eta_1(x)\,dx \quad [1<m<N] \quad (10)$$

$$Q_{1,o}(o,t) = \int_o^\infty Q_{2,o}(x,t)\,\eta_1(x)\,dx + N\lambda_1 P_o(t) \tag{11}$$

$$U_n(o,t) = \int_o^\infty U_{n+1}(y,t)\,\eta_2(y)\,dy + \int_o^\infty Q_{1,n}(x,t)\,\eta_1(x)\,dx \tag{12}$$

and

$$U_1(o,t) = \int_o^\infty U_2(y,t)\,\eta_2(y)\,dy + \int_o^\infty Q_{1,1}(x,t)\eta_1(x)\,dx + \lambda_2 P_o(t) \tag{13}$$

The equations (6) and (9) arise because the priority units come from the finite population and it is not possible to have N priority units in the system just after the service completion of a priority unit.

SOLUTION OF THE MODEL

We will now derive the solution of the above equations under steady state by employing the generating functions and the finite transforms. We shall not attempt here to give the rigorous proof that the stationary distribution exists whatever may be the initial condition. We shall later on show that the criterion for the non-saturation is given by

$$\rho_2\left(1 + N\rho_1 \sum_{l=o}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l}\right) < 1$$

where $\rho_i = \lambda_i \int_o^\infty x\, S_i(x)\,dx$ $(i=1, 2)$ and ϕ_l is defined in the equation (54). We will drop the argument t in the state probabilities mentioned above to denote the probabilities under steady state. Thus for example, the steady state probability of $P_{m,n}(x,y,t)\,dx\,dy$ will be denoted as $P_{m,n}(x,y)\,dx\,dy$. Further, let us introduce the following generating functions of steady state probabilities:

$$f_m(x,y,\alpha) = \sum_{n=1}^\infty P_{m,n}(x,y)\,\alpha^n\,;\; g_m(x,\alpha) = \sum_{n=o}^\infty Q_{m,n}(x)\,\alpha^n$$

$$\text{and } J(y,\alpha) = \sum_{n=1}^\infty U_n(y)\,\alpha^n$$

Thus the equations (1) to (3) after employing the generating functions, can be written as

$$\left\{\frac{\partial}{\partial x} + [(N-m)\lambda_1 + \lambda_2(1-\alpha) + \eta_1(x)]\right\} f_m(x,y,\alpha) = (N-m+1)\lambda_1 f_{m-1}(x,y,\alpha), \tag{14}$$

$$\left\{ \frac{\partial}{\partial x} + [(N - m)\lambda_1 + \lambda_2 (1 - a) + \eta_1 (x)] \right\} g_m (x, \alpha)$$
$$= (N - m + 1) \lambda_1 g_{m-1} (x, a) \quad (15)$$

$$\left\{ \frac{\partial}{\partial y} + [N\lambda_1 + \lambda_2 (1 - a) + \eta_2 (y)] \right\} J(y,\alpha) = \int_0^\infty f_1 (x,y,\alpha)\, \eta_1(x)\, dx \quad (16)$$

Let the following finite discrete transform be employed

$$A_m (x,y,\alpha) = \sum_{j=m}^{N-1} \binom{j}{m} f_{N-j} (x, y, \alpha) \quad [o \leqslant m \leqslant (N-1)] \quad (17)$$

and

$$B_m (x, \alpha) = \sum_{j=m}^{N-1} \binom{j}{m} g_{N-j} (x, \alpha) \quad [o \leqslant m \leqslant (N-1) \quad (18)$$

The inverse transforms of (17) and (18) are seen to be

$$f^l (x, y, \alpha) = \sum_{K=o}^{l-1} (-)^K \binom{N-l+K}{K} A_{N-l+K} (x, y, \alpha) \; [1 \leqslant l \leqslant N] \quad (19)$$

and

$$g_l (x, \alpha) = \sum_{K=o}^{l-1} (-)^K \binom{N-l+K}{K} B_{N-l-K} (x, \alpha) \quad [1 \leqslant l \leqslant N] \quad (20)$$

Now changing m to $(N-r)$ in (14) and (15), multiplying both sides by ${}^rC_m$ and summing from $r = m$, to $r = (N-1)$ we obtain

$$\left\{ \frac{\partial}{\partial x} + [m\lambda_1 + \lambda_2 (1 - \alpha) + \eta_1 (x)] \right\} A_m (x, y, \alpha) = 0 \quad (21)$$

and

$$\left\{ \frac{\partial}{\partial x} + [m\lambda_1 + \lambda_2 (1 - \alpha) + \eta_1 (x)] \right\} B_m (x, \alpha) = 0 \quad (22)$$

On integrating the equations (21) and (22) from o to x, we get

$$A_m (x, y, \alpha) = A_m (o, y, \alpha) \exp \left\{ -[m\lambda_1 + \lambda_2 (1 - \alpha)]\, x - \int_o^x \eta_1(x) dx \right\} \quad (23)$$

and

$$B_m (x, \alpha) = B_m (o, \alpha) \exp \left\{ -[m\lambda_1 + \lambda_2 (1 - \alpha)]\, x - \int_o^x \eta_1 (x)\, dx \right\} \quad (24)$$

Similarly the boundary conditions (6) to (11) on employing the generating functions give the following equations :

$$f_n(o, y, \alpha) = 0 \tag{25}$$

$$f_m(o, y, \alpha) = \int_o^\infty f_{m+1}(x, y, \alpha)\, \eta_1(x)\, dx \qquad [1 < m < N] \tag{26}$$

$$f_1(o, y, \alpha) = \int_o^\infty f_2(x, y, \alpha)\, \eta_1(x)\, dx + N\lambda_1 J(y, \alpha) \tag{27}$$

$$g_N(o, \alpha) = 0 \tag{28}$$

$$g_m(o, \alpha) = \int_o^\infty g_{m+1}(x, \alpha)\, \eta_1(x)\, dx \qquad [1 < m < N] \tag{29}$$

and

$$g_1(o, \alpha) = \int_o^\infty g_2(x, \alpha)\, \eta_1(x)\, dx + N\lambda_1^* P_o \tag{30}$$

We obtain, employing the transform in (17) to the equations (25)—(27)

$$A_m(o, y, \alpha) = \left\{ \int_o^\infty [\, A_{m-1}(x, y, \alpha) + A_m(x, y, \alpha)\,]\, \eta_1(x)\, dx \right.$$

$$\left. + \binom{N-1}{m} N \lambda_1 J(y, \alpha) - \binom{N}{m} \int_o^\infty A_{N-1}(x, y, \alpha)\, \eta_1(x)\, dx \right\} \tag{31}$$

Substituting (23) in (31), we get

$$\bar{A}_m(o, y, \alpha) = \left\{ \left(1 - \bar{S}_1[\, m\lambda_1 + \lambda_2(1-\alpha)\,] \right) \right\}^{-1} .$$

$$\cdot \left\{ A_{m-1}(o,y,\alpha)\, \bar{S}_1[(m-1)\lambda_1 + \lambda_2(1-\alpha)] - \binom{N}{m} A_{N-1}(o, y, \alpha)\, \bar{S}_1[(N-1)\lambda_1 \right.$$

$$\left. + \lambda_2(1-\alpha)] + N\lambda_1 J(y, \alpha) \binom{N-1}{m} \right\} \tag{32}$$

where $\bar{S}_i(s) = \int_o^\infty e^{-sx} S_i(x)\, dx \qquad [Re(s) \geq o\,; i = 1, 2]$

Let us now define

$$C_m'(\alpha, s) = \prod_{l=1}^{m} \left(\frac{\bar{S}_1 [(l-1)\lambda_1 + \lambda_2 (1-\alpha) + s]}{1 - \bar{S}_1 [l\lambda_1 + \lambda_2 (1-\alpha) + s]} \right) \tag{33}$$

$$[1 \leqslant m \leqslant (N-1)]$$

and

$$C_m(\alpha, s) = \prod_{l=o}^{m} \left(\frac{\bar{S}_1 [l\lambda_1 + \lambda_2 (1-\alpha) + s]}{1 - \bar{S}_1 [l\lambda_1 + \lambda_2 (1-\alpha) + s]} \right) \tag{34}$$

$$[0 \leqslant m \leqslant (N-1)]$$

The relations between $C_m'(\alpha, s)$ and $C_m(\alpha, s)$ are given by

$$\frac{C_m'(\alpha, s)}{1 - \bar{S}_1 [\lambda_2 (1-\alpha) + s]} = \frac{C_{m-1}(\alpha, s)}{1 - \bar{S}_1 [m\lambda_1 + \lambda_2 (1-\alpha) + s]}$$

$$= \frac{C_m(\alpha, s)}{\bar{S}_1 [m\lambda_1 + \lambda_2 (1-\alpha) + s]} \tag{35}$$

Dividing both sides of (32) by $C_m'(\alpha, o)$ and using the relations in (35) we get

$$\frac{A_m(o, y, \alpha)}{C_m'(\alpha, o)} = \left\{ \frac{A_{m-1}(o, y, \alpha)}{C_{m-1}'(\alpha, o)} + \frac{N\lambda_1 J(y, \alpha)}{1 - \bar{S}_1 [\lambda_2 (1-\alpha)]} \binom{N-1}{m} \frac{1}{C_{m-1}(\alpha, o)} \right.$$

$$\left. - \frac{A_{N-1}(o, y, \alpha)\, \bar{S}_1 [(N-1)\lambda_1 + \lambda_2 (1-\alpha)]}{1 - \bar{S}_1 [\lambda_2 (1-\alpha)]} \binom{N}{m} \frac{1}{C_{m-1}(\alpha, o)} \right\} \tag{36}$$

Changing m to $(m-1)$, $(m-2)$..............1 successively and adding we get

$$\frac{A_m(o, y, \alpha)}{C_m'(\alpha, o)} = \left\{ \frac{A_o(o, y, \alpha)}{C_o'(\alpha, o)} + \frac{N\lambda_1 J(y, \alpha)}{1 - \bar{S}_1 [\lambda_2 (1-\alpha)]} \sum_{l=1}^{m} \binom{N-1}{l} \frac{1}{C_{l-1}(\alpha, o)} \right.$$

$$\left. - \frac{A_{N-1}(o, y, \alpha)\, \bar{S}_1 [(N-1)\lambda_1 + \lambda_2 (1-\alpha)]}{1 - \bar{S}_1 [\lambda_2 (1-\alpha)]} \sum_{l=0}^{m} \binom{N}{l} \frac{1}{C_{l-1}(\alpha, o)} \right. \tag{37}$$

Now adding all the equations (25), (26) and (27) one can easily see

$$A_o\,(o, y, \alpha) = \int_o^\infty A_o\,(x, y, \alpha)\eta_1(x)\,dx - \int_o^\infty A_{N-1}\,(x, y, \alpha)\,\eta_1(x)\,dx + N\lambda_1 J\,(y, \alpha) \tag{38}$$

Substituting (23) in (38), we get

$$A_o\,(o, y, \alpha) = \frac{1}{1-\bar{S}_1[\lambda_2(1-\alpha)]}\Big\{N\lambda_1 J(y,\alpha) - A_{N-1}\,(o, y, \alpha)\,\bar{S}_1[(N-1)\lambda_1 + \lambda_2(1-\alpha)]\Big\} \tag{39}$$

Defining $C_o'\,(\alpha, s) = C_{-1}\,(\alpha, s) = 1$ and substituting (39) in (37) we get after a little manipulation

$$\frac{A_m(o, y, \alpha)}{C_m'(\alpha, o)} = \frac{1}{1-\bar{S}_1[\lambda_2(1-\alpha)]}\Big\{N\lambda_1\,J\,(y, \alpha)\sum_{l=o}^{m}\binom{N-1}{l}\frac{1}{C_{l-1}\,(\alpha, o)} - A_{N-1}\,(o, y, \alpha)\,\bar{S}_1[(N-1)\lambda_1+\lambda_2\,(1-\alpha)]\sum_{l=o}^{m}\binom{N}{l}\frac{1}{C_{l-1}\,(\alpha, o)}\Big\} \tag{40}$$

Putting $m = (N-1)$ in (40) and rearranging the terms we obtain

$$A_{N-1}\,(o, y, \alpha) = \frac{N\lambda_1\,J\,(y, \alpha)}{\bar{S}_1\,[(N-1)\,\lambda_1 + \lambda_2\,(1-\alpha)]}\,G\,(\alpha, o) \tag{41}$$

where

$$G\,(\alpha, s) = \frac{\displaystyle\sum_{l=o}^{N-1}\binom{N-1}{l}\frac{1}{C_{l-1}\,(\alpha, s)}}{\displaystyle\sum_{l=o}^{N}\binom{N}{l}\frac{1}{C_{l-1}\,(\alpha, s)}} \tag{42}$$

Thus we get

$$A_m\,(o, y, \alpha) = \frac{N\,\lambda_1\,J\,(y, \alpha)\,C_{m-1}\,(\alpha, o)}{1-\bar{S}_1\,[m\lambda_1 + \lambda_2\,(1-\alpha)]} \cdot \left\{\sum_{l=o}^{m}\binom{N-1}{l}\frac{1}{C_{l-1}\,(\alpha, o)} - G\,(\alpha,o)\sum_{l=o}^{m}\binom{N}{l}\frac{1}{C_{l-1}\,(\alpha, o)}\right\} \tag{43}$$

Similar procedure can be adopted with the boundary conditions (28), (29) and (30) and we obtain

$$B_m(o,a) = \left[[N\lambda_1^* P_o C_{m-1}(a,o) / 1 - \bar{S}_1 [m\lambda_1 + \lambda_2(1-a)\} \right]$$

$$\cdot \left[\sum_{l=o}^{m} \binom{N-1}{l} \frac{1}{C_{l-1}(a,o)} - G(a,o) \sum_{l=o}^{m} \binom{N}{l} \frac{1}{C_{l-1}(a,o)} \right] \quad (44)$$

Now, the equation (16) can be written as

$$\left\{ \frac{\partial}{\partial y} + [N\lambda_1 + \lambda_2(1-a) + \eta_2(y)] \right\} J(y,a) = \int_o^\infty A_{N-1}(x,y,\alpha)\eta_1(x)dx \quad (45)$$

Substituting the value of $A_{N-1}(x, y\ a)$ from (23) integrating and using (41) the equation (45) can be written as

$$\left\{ \frac{\partial}{\partial y} + [N\lambda_1 \{1 - G(a,o)\} + \lambda_2(1-a) + \eta_2(y)] \right\} J(y,a) = 0 \quad (46)$$

The solution of (46) is given by

$$J(y,a) = J(o,a) \exp \left\{ -[N\lambda_1 \{1 - G(a,o)\} + \lambda_2(1-a)]\, y - \int_o^y \eta_2(y)\, dy \right\} \quad (47)$$

The equations (12) and (13) when multiplied by appropriate powers of a and added give after employing the relation (4),

$$J(o,\alpha) = \left\{ \frac{1}{a} \int_o^\infty J(y,a)\eta_2(y)\, dy + \int_o^\infty B_{N-1}(x,a)\, \eta_1(x)\, dx \right.$$

$$\left. - [N\lambda_1^* + \lambda_2(1-a)]\, P_o \right\} \quad (48)$$

With the help of (24), (44) and (47), the equation (48) simplifies to

$$J(o,a) = \frac{-[N\lambda_1^* + \lambda_2(1-a)]\, P_o}{1 - \frac{1}{a} \bar{S}_2 [N\lambda_1(1 - G(a,o) + \lambda_2(1-a)]} \quad (49)$$

The generating function of the state probabilities is defined by

$$\Pi(\alpha, \beta) = \Big\{ P_o + \int_o^\infty J(y, \alpha)\, dy + \sum_{m=1}^{N} \beta^m \int_o^\infty g_m(x, \alpha)\, dx + \sum_{m=1}^{N} \beta^m \int_o^\infty \int_o^\infty f_m(x, y, \alpha)\, dx\, dy \Big\} \tag{50}$$

After a little bit of simplification, (50) can be written as

$$\Pi(\alpha, \beta) = P_\circ \Big\{ \frac{(\alpha - 1)\overline{S}_2 [N\lambda_1 \{1 - G(\alpha,0)\} + \lambda_2(1 - \alpha)]}{\alpha - \overline{S}_2 [N\lambda_1 \{1 - G(\alpha, 0)\} + \lambda_2(1 - \alpha)]} + \frac{N \sum_{m=1}^{N} \beta^m \sum_{K=0}^{m-1} (-)^K \binom{N - m + K}{K} T_{N-m+K}(\alpha)}{\alpha - \overline{S}_2[N\lambda_1\{1 - G(\alpha,0)\} + \lambda_2(1 - \alpha)]} \Big[(\lambda_1^* - \lambda_1)\alpha + \overline{S}_2[N\lambda_1\{1 - G(\alpha,0)\} + \lambda_2(1 - \alpha)](\lambda_1\alpha - \lambda_1^*) \Big] \Big\} \tag{51}$$

where

$$T_r(\alpha) = \frac{C_{r-1}(\alpha, 0)}{r\lambda_1 + \lambda_2(1 - \alpha)} \Big\{ \sum_{l=0}^{r} \binom{N-1}{l} \frac{1}{C_{l-1}(\alpha,0)} - G(\alpha,0) \sum_{l=0}^{r} \binom{N}{l} \frac{1}{C_{l-1}(\alpha,0)} \Big\} \tag{52}$$

The $P_\circ$ is determined such that $\pi(1, 1) = 1$ · Thus we have

$$P_\circ = \frac{1 - \lambda_2\eta_2 \Big(1 + N\lambda_1\eta_1 \sum_{l=0}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l}\Big)}{1 + N\lambda_1^*\eta_1 \sum_{l=0}^{N-1} \binom{N-1}{l} \frac{1}{\varphi_l}} \tag{53}$$

where

$$\phi_l = \begin{cases} 1 & \text{if } l = 0 \\ \prod_{K=1}^{l} \left(\dfrac{\overline{S}_1[K\lambda_1]}{1 - \overline{S}_1[K\lambda_1]} \right) & \text{if } 1 \leqslant l < N \end{cases} \tag{54}$$

and $\eta_i = \int_o^\infty x\, S_i(x)\, dx \qquad (i = 1, 2).$

The mean queue length of priority and non-priority can be obtained from (51) by the process of differentiation.

PARTICULAR CASES

We shall discuss some of the special cases of the model discussed above.

(a) $\lambda_1 = \lambda_1^*$: In this case we assume that the priority units arrive even when the server is free. The generating function (51) and the value of $P_\circ$ reduce respectively to

$$\Pi(\alpha,\beta) = \Bigg[P_\circ \frac{(\alpha - 1)S_2[N\lambda_1(1 - G(\alpha,0)) + \lambda_2(1 - \alpha)]}{\alpha - \bar{S}_2[N\lambda_1(1 - G(\alpha,0)) + \lambda_2(1 - \alpha)]} \cdot \left\{1 + N\lambda_1 \sum_{m=1}^{N} \beta^m \sum_{K=0}^{m-1} \binom{N-m+K}{K} T_{N-m+K}(\alpha) \right\}\Bigg] \tag{55}$$

and

$$P_\circ = \left\{ \left\{1 - \lambda_2\eta_2 \left(1 + N\lambda_1\eta_1 \sum_{l=0}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l}\right)\right\} \Big/ \left\{1 + N\lambda_1\eta_1 \sum_{l=0}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l}\right\}\right\} \tag{56}$$

(b) $\lambda_1^* = 0$: In this case the priority units arrive only when there are ordinary customers in the system. Thus for this case (51) and (53) reduce to

$$\Pi(\alpha, \beta) = P_o \left\{ 1 - \frac{\alpha [1 - \bar{S}_2 [N\lambda_1 (1 - G(\alpha, o)) + \lambda_2 (1 - \alpha)]]}{\alpha - \bar{S}_2 [N\lambda_1 (1 - G(\alpha, o)) + \lambda_2 (1 - \alpha)]} \cdot \left(1 + N\lambda_1 \sum_{m=1}^{N} \beta^m \sum_{K=o}^{m-1} (-)^k \binom{N-m+k}{K} T_{N-m+V}(\alpha) \right)\right\} \tag{57a}$$

and

$$P_o = \left[1 - \lambda_2 \eta_2 \left(1 + N \lambda_1 \eta_1 \sum_{l=o}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l} \right)\right] \tag{57b}$$

respectively.

(c) when $N = 1$, the model reduces to the breakdown model considered by the author.[9] The expression (51) reduces to

$$\Pi (\alpha, \beta) = \frac{1 - \lambda_2 \eta_2 (1 + \lambda_1 \eta_1)}{1 + \lambda_1^* \eta_1} \left\{\left(1 + \lambda_1^* \frac{1 - \bar{S}_1 [\lambda_2 (1 - \alpha)]}{\lambda_2 (1 - \alpha)} \right) - \frac{\alpha (1 - \bar{S}_2 [\lambda_1 (1 - \bar{S}_1 [\lambda_2 (1 - \alpha)]) + \lambda_2 (1 - \alpha)])}{\alpha - \bar{S}_2 [\lambda_1 (1 - \bar{S}_1 [\lambda_2 (1 - \alpha)]) + \lambda_2 (1 - \alpha)]} \left(1 + \lambda_1 \frac{1 - S_1[\lambda_2(1-\alpha)]}{\lambda_2 (1 - \alpha)} \right)\right\} \tag{58}$$

(*d*) If we take the limit of $\pi(\alpha, \beta)$ as α tends to 1, after putting $\lambda_1 = \lambda_1^*$, we obtain the results of the simple machine intereference problem. It is easy to observe that

$$\lim_{\alpha \to 1} T_r(\alpha) = \beta_r = \frac{\phi_{r-1}}{r} \sum_{l=r+1}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l} \tag{59}$$

Hence

$$\Pi(1, \beta) = \frac{1}{1 + N\lambda_1 \eta_1 \sum_{l=o}^{N-1} \binom{N-1}{l} \frac{1}{\phi_l}} \left(1 + N\lambda_1 \sum_{M=1}^{N} \beta^m \sum_{K=o}^{m-1} (-)^K \binom{N-m+K}{K} \beta_{N-m+K}\right) \tag{60}$$

– a result which is obtained by Takacs.[11]. It is evident from (60) that the priority queue is not affected by the preemptive priority discipline. Thus the machine availability will be the same for the model considered in this paper.

(*e*) When $N = 0$ or $\lambda_1 = \lambda_1^* = 0$ the results (51) and (53) correspond to the classical single server queuing process.

STOCHASTIC LAW OF THE BUSY PERIODS

Here we shall investigate the distribution of the length of busy periods for the model considered. A busy period is the length of the time during which the server is occupied. For the process considered, the busy period either begins with an arrival of a priority unit or begins with the arrival of an ordinary customer. Thus we separately investigate the time period during which the server is busy in servicing the units when

(*a*) the time period begins with an arrival of priority unit and

(*b*) the time period begins with an arrival of ordinary unit.

In order to determine the busy period distribution for the case (*a*) we proceed as follows: The busy period begins with an arrival of a priority unit. Then we consider the tour of the particle in the state phase space and the process stops as soon as the particle reaches empty state at time t. If $\gamma_1(t)$ denotes the busy period density and $P'_o(t)$ denotes the probability that the system is empty at time t, then

$$\gamma_1(t) = \frac{d}{dt} P'_o(t) \tag{61}$$

For this process, the equations (1), (2) and (3) are valid and further we have the following equations:

$$\frac{d}{dt} P'_o(t) = \int_o^\infty U_1(y,t)\, \eta_2(y)\, dy + \int_o^\infty Q_{1,n}(x,t)\, \eta_1(x)\, dx \tag{62}$$

$$P_{N,n}(o, y, t) = 0 \qquad \text{for all } n \tag{63}$$

$$P_{m,n}(o, y, t) = \int_{o}^{\infty} P_{m+1,n}(x, y, t)\, \eta_1(x)\, dn + \delta_{1,m}\, U_n(y, t) \tag{64}$$

$$Q_{N,n}(o,t) = 0 \text{ for all } n \tag{65}$$

$$Q_{m,n}(o,t) = \int_{o}^{\infty} Q_{m+1,n}(x, t)\, \eta_1(x)\, dx \tag{66}$$

$$U_n(o,t) = \int_{o}^{\infty} U_{n+1}(y,t)\, \eta_2(y)\, dy + \int_{o}^{\infty} Q_{1,n}(x,t)\, \eta_1(x)\, dx \tag{67}$$

and $$Q_{1,o}(x,o) = \delta(x + o) \tag{68}$$

where $\delta(x)$ is the Dirac delta function and δ_{ij} is the usual Kronecker delta symbol.

Taking the Laplace transform of the equations (1), (2) and (3) with the boundary condition (68), we get

$$\left\{\frac{\partial}{\partial x} + \left[(N-m)\lambda_1 + \lambda_2 + s + \eta_1(x)\right]\right\}\bar{P}_{m,n}(x, y, s)$$

$$= (N - m + 1)\lambda_1 \bar{P}_{m-1,n}(x, y, s) + \lambda_2 \bar{P}_{m,n-1}(x, y, s) \tag{69}$$

$$\left\{\frac{\partial}{\partial x_o} + \left[(N-m)\lambda_1 + \lambda_2 + s + \eta_1(x)\right]\right\}\bar{Q}_{m,n}(x, s) - \delta(x)\,\delta_{n,o}$$

$$= (N - m + 1)\lambda_1 \bar{Q}_{m-1,n}(x, s) + \lambda_2 \bar{Q}_{m,n-1}(x, s) \tag{70}$$

$$\left\{\frac{\partial}{\partial y} + \left[N\lambda_1 + \lambda_2 + s + \eta_2(y)\right]\right\}\bar{U}_n(y, s)$$

$$= \lambda_2 \bar{U}_{n-1}(y, s) + \int_{o}^{\infty} \bar{P}_{1,n}(x, y, s)\, \eta_1(x)\, dx \tag{71}$$

where

$$\bar{P}_{m,n}(x, y, s) = \int_{o}^{\infty} e^{-st} P_{m,n}(x, y, t)\, dt \text{ etc.}$$

which are convergent whenever $Re(s) \geqslant 0$.

The equation (69), (70) and (71) and the boundary conditions are of the similar type as before and accordingly the problem can be solved by adopting the similar procedure. Thus we have

$$\bar{J}(o, \alpha, s) = \frac{[G(\alpha, s) - s\, P_o'(s)]\,\alpha}{\alpha - \bar{S}_2[N\lambda_1(1 - G(\alpha, s)) + \lambda_2(1 - \alpha) + s]} \tag{72}$$

Let α_s be the root of the equation in α,

$$\alpha = \bar{S}_2[N\lambda_1(1 - G(\alpha, s)) + \lambda_2(1 - \alpha) + s] \tag{73}$$

which lies inside $|\alpha|=1$. Thus from (61) and (72) we get the Laplace transform of the busy period distribution starting with a periority unit and is given by

$$s\,\bar{P}'_{o}(s) = \gamma_1(s) = G(\alpha_s, s) \tag{74}$$

Now we shall investigate the length of busy periods which will begin with the arrival of an ordinary unit. Since $Q_{m,n}(x,t)$ will not exist for this problem, we only take the equation (1) and (3) and also impose the condition that the system stops as soon as empty state is reached. If $P_o(t)$ denote the probability that the system is empty at time t, then the density of the busy period distribution is given by

$$\gamma_2(t) = \frac{d}{dt} P_o(t) \tag{75}$$

The additional equation for the process is

$$\frac{d}{dt} P_o(t) = \int_o^\infty U_1(y,t)\,\eta_2(y)\,dy \tag{76}$$

together with the boundary conditions (6) (7) (8) and

$$U_n(o,t) = \int_o^\infty U_{n+1}(y,t)\,\eta_2(y)\,dy \tag{77}$$

The initial condition is

$$U_n(y,o) = \delta(y+\delta)\,\delta_{1,n} \tag{78}$$

where $\delta(y+)$ is the Dirac delta function.

As before by employing Laplace transform, we obtain

$$\left\{\frac{\partial}{\partial x} + \left[(N-m)\lambda_1 + \lambda_2 + s + \eta_1(x)\right]\right\}\bar{P}_{m,n}(x,y,s)$$

$$= (N-m+1)\lambda_1 + \bar{P}_{m-1,n}(x,y,s) + \lambda_2\,\bar{P}_{m,n-1}(x,y,s) \tag{79}$$

$$\left\{\frac{\partial}{\partial y} + \left[N\lambda_1 + \lambda_2 + s + \eta_2(y)\right]\right\}\bar{U}_n(y,s)$$

$$= \lambda_2\,\bar{U}_{n-1}(y,s) + \int_o^\infty \bar{P}_{1,n}(x,y,s)\,\eta_1(x)\,dx \tag{80}$$

The equations can be solved as before and finally we get

$$(o,\alpha,s) = \frac{\bar{S}_2\left[N\lambda_1(1-G(\alpha,s)) + \lambda_2(1-\alpha) + s\right] - s\,\bar{P}_o(s)}{1 - \frac{1}{\alpha}\,\bar{S}_2\left[N\lambda_1(1-G(\alpha,s)) + \lambda_2(1-\alpha) + s\right]} \tag{81}$$

Since $\bar{J}(o,\alpha,s)$ is analytic for $Re(s)\geq o$ and $|\alpha|\leqslant 1$ the numerator of (81) should vanish for $\alpha = \alpha_s$ where α_s is the root of the equation (73) inside $|\alpha| = 1$. Thus we obtain:

$$s\,\bar{P}_o(s) = \bar{\gamma}_2(s) = \alpha_s \tag{82}$$

Now since the arrival rate of each of the priority units, and of the ordinary units are Poisson we observe that with probability $[N\lambda_1^*/N\lambda_1^*+\lambda_2]$ the busy period starts with a priority, unit and with probability $[\lambda_2/N\lambda_1^* + \lambda_2]$ the busy period starts with an ordinary unit. Thus, the Laplace transform, of the busy period distribution $r(t)$ is given by

$$\bar{\gamma}(s) = \frac{N\lambda^*_1}{N\lambda_1^* + \lambda_2} G(a_s \quad s) + \frac{\lambda_2}{N\lambda_1^* + \lambda_2} a_s \tag{83}$$

when $N = 0$, this result corresponds to the classical queuing process and when $N = 1$ the result agrees with the result of Gaver[8] and Keilson[9].

The inversion of (83) is, however, difficult even for the simplest cases. But the moments of the length of the busy period can be obtained directly by successive differentiation of (83). Thus, the expected length of the busy period is given by

$$\int_0^\infty t\,\gamma(t)\,dt =$$

$$\frac{N\lambda_1^*\eta_1 \sum_{l=0}^{N-1}\binom{N-1}{l}\frac{1}{\phi_l} + \lambda_2\eta_2\left(1 + N\lambda_1\eta_1 \sum_{l=0}^{N-1}\binom{N-1}{l}\frac{1}{\phi_l}\right)}{\left(N\lambda^*_1 + \lambda_2\right)\left[1 - \lambda_2\eta_2\left(1 + N\lambda_1\eta_1\sum_{l=0}^{N-1}\binom{N-1}{l}\frac{1}{\phi_l}\right)\right]} \tag{84}$$

and similarly other moments can be calculated.

DISCUSSION

We have already stated the condition under which the steady state solution for the model exists. The condition is obtained by assuming the steady state probability P_0 which is given in (53) should be non-negative.

It may be interesting to note that the arrival rate of priority units when the system is empty, has no influence for the existence of steady state solution even though it is different from the arrival rate of the priority units when it is occupied. Thus, the criterion for the existence of steady state solution will be the same whether priority units arrive or do not arrive when the system is empty.

It may be remarked that the Laplace transform of the time dependent probability generating function can be obtained by using a similar method. However, the inversion is difficult and therefore the derivation is omitted.

Further it may be observed that we can introduce the other type of priority disciplines for this model. For example one can introduce the head-of-the-line priority discipline[1] in which the priority unit if it arrives during the service period of an ordinary unit should wait till the ordinary unit in the facility completes a service. Such changes in queue discipline affects the results considerably.

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