ON THE CHANGE OF PHASE WHEN BOTH THE PHASES HAVE DIFFERENT THERMAL PROPERTIES

K. L. AHUJA

Defence Science Laboratory, Delhi 6

ABSTRACT

The growth of a plane boundary advancing in the supercooled fluid is discussed. The solid and the fluid have both been assumed to be of finite conductivity and different densities. The assumption of difference in densities results in a convective motion of the fluid and the mass solidified per unit area per unit time will occupy, on solidification, a different volume from that originally occupied by the fluid. The effect of finite conductivity of the solid on the growth has been studied by taking the Prandtl number different from unity. It is shown that the equivalence of interface temperature and the surface temperature of the solid leads to the results discussed by Chambre (1).

INTRODUCTION

A most important class of problems is that in which a substance changes from one state to another with emission or absorption of heat. Such problems occur in many contexts most important of which are the solidification or melting of ice. A physically restricted form of the problem was first solved by Neumann, and Stefan, Ingersoll and Zobel and others carried out the subsequent extensions. But in all such investigations, a physical limitation was imposed in that, that the densities of both the media were assumed to be equal and the effects of convective motion were ignored. Chambre, however, took into account the effects of density differences and solved the problem by assuming the solid to be of infinite conductivity.

It is the purpose of this paper to study the growth of a plane solid advancing in the surrounding of a supercooled fluid. The solid and the fluid have been taken to be of finite thermal conductivity and of unequal densities. The effect of finite thermal conductivity of solid on the growth rate, by assuming the Prandtl number to be different from unity has been studied and the results have been exhibited graphically. It is also shown that in the event of interface temperature becoming equal to the surface temperature, the resulting equation in (3·15) reduces to equation (22) of Chambre for a plane case.

DERIVATION OF EQUATIONS

Since both the solid and the fluid are assumed to be of finite thermal conductivity, the equation of the energy flow in the solid and the momentum and energy equations in the fluid together with the boundary conditions are given by the following.

$$\frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial r^2} \qquad o < r < R(t) \quad \ldots \quad \ldots \quad (2\cdot 1)$$

$$\left.\begin{array}{ll} T_2 = T_A & ; \quad r = o \\ T_2 = T_S & ; \quad r = R(t) \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

and

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} = \nu\,\frac{\partial^2 u}{\partial r^2} \quad \cdots \quad (2\cdot3)$$

$$\frac{\partial T_1}{\partial t} + u\,\frac{\partial T_1}{\partial r} = a_1\,\frac{\partial^2 T_1}{\partial r^2} \qquad R\,(t) < r < \infty \quad \cdots \quad (2\cdot4)$$

$$\left.\begin{array}{ll} T_1 = T_S & ;\ r = R(t) \\ \left.\begin{array}{l} T_1 = T_\infty \\ u = o \end{array}\right\} & ;\ r = \infty \end{array}\right\} \quad \cdots \quad (2\cdot5)$$

where ν is the kinematic viscosity of the fluid and 'α' is the thermal diffusivity. Subscripts 1 and 2 stand respectively for the fluid and the solid. R (t) specifies the position of the solidification front.

The surface of separation moves with a velocity $\frac{dR}{dt}$ and the mass of solid $\rho_2\,\frac{dR}{dt}$ per unit area per unit time so formed is derived from a mass flow of the amount $\rho_1\left(\frac{dR}{dt} - u\right)$. From the principle of conservation of mass

$$\rho_2\,\frac{dR}{dt} = \rho_1\left(\frac{dR}{dt} - u\right)$$

$$or\ u = -\,\frac{\rho_2 - \rho_1}{\rho_1}\,\frac{dR}{dt} \quad \cdots \quad (2\cdot6)$$

Denoting $\frac{\rho_2 - \rho_1}{\rho_1}$ by ϵ, (2·6) can be written as $u = -\,\epsilon\,\frac{dR}{dt}$ .. (2·7)

Also the boundary condition at the surface of separation needs that the amount of heat liberated per unit area per unit time must be removed by conduction. This requires that

$$\frac{\lambda_2}{\lambda_1}\,\frac{\partial T_2}{\partial r} - \frac{\partial T_1}{\partial r} = \frac{1+\epsilon}{a_1}\,\theta\,\frac{dR}{dt} \quad \cdots \quad (2\cdot8)$$

where λ and ρ are the thermal conductivity and density and $\theta = \frac{L}{C_1} + T_s\left(\frac{C_2}{C_1} - 1\right)$,

L is the latent heat and C is the specific heat.

Equations (2·1) to (2·8) are the equations of the problem to be solved.

PARTICULAR SOLUTION OF THE PROBLEM

Because of the non-linearity of the problem due to the presence of the unknown $R(t)$, it is difficult to find a general solution under the given boundary conditions. We, therefore, seek a particular solution by taking R, the position of the moving front, to be proportional to $\sqrt{t}$ and write it as

$$R = 2\beta\sqrt{\nu t} \quad \ldots \quad (3\cdot1)$$

where β is a positive number to be determined.

A. *Solution for the solid region*

A particular solution of equation (2·1) is given by the following

$$T_2 = A_2 + B_2 \, erf \, \frac{r}{2\sqrt{a_2 t}} \quad \ldots \quad (3\cdot2)$$

where A_2 and B_2 are constants to be determined from the boundary conditions (2·2) The boundary condition at $r=0$ gives $A_2 = T_A$ and we have

$$T_2 = T_A + B_2 \, erf \, \frac{r}{2\sqrt{a_2 t}} \quad \ldots \quad (3\cdot3)$$

The other boundary condition at $r = R(t)$ gives $T = T_S$ and we get

$$T_S = T_A + B_2 \, erf \, \beta\sqrt{\frac{\nu}{a_2}} \quad \ldots \quad (3\cdot4)$$

Also

$$\frac{\partial T_2}{\partial r}\bigg|_{r = R(t)} = \frac{2}{\sqrt{\pi}} \, \frac{B_2}{2\sqrt{a_2 t}} \, e^{-\frac{\beta^2 \nu}{\sqrt{a_2}}} \quad \ldots \quad (3\cdot5)$$

B. *Solution for the fluid region.*

Following Chambre, the solution of equations (2·3) and (2·4) can be found by using the following similarity transformation on velocity and temperature

$$u = 2\sqrt{\frac{\nu}{\pi t}} \, f(\eta) \quad \ldots \quad (3\cdot6)$$

$$T_1 = T(\eta) \quad \ldots \quad (3\cdot7)$$

$$\eta = \frac{r}{2\sqrt{\nu t}} \quad \ldots \quad (3\cdot8)$$

Substituting for u from (3·6) in (2·3), we get the following differential equation for $f(\eta)$

$$\tfrac{1}{2} f'' - \frac{2}{\sqrt{\pi}} ff' + \eta f' + f = 0 \quad \ldots \quad (3\cdot9)$$

where dashes denote differentiation with respect to η

The solution of (3·9) is given by

$$f = - e^{-\eta^2}\left[\frac{1}{\epsilon\beta}\left\{\frac{2}{\sqrt{\pi}}\, e^{-\beta^2} - \epsilon\beta\, erf\beta\right\} + erf\eta\right]^{-1} \quad \ldots \quad (3·10)$$

$$\text{or} \quad u = - 2\sqrt{\frac{\nu}{\pi t}} \; \frac{\epsilon\beta e^{\beta^2} e^{-\eta^2}}{\frac{2}{\sqrt{\pi}} + \epsilon\beta e^{\beta^2}(erf\eta - erf\beta)} \quad \ldots \quad (3·11)$$

To obtain the solution of (2·4) we substitute for T_1, from (3·7) in (2·4). The resulting differential equation in T is given by the following form.

$$T'' + \sigma\left[2\eta + \frac{4\epsilon\beta e^{\beta^2}}{\sqrt{\pi}} \; \frac{e^{-\eta^2}}{\frac{2}{\sqrt{\pi}} + \epsilon\beta e^{\beta^2}(erf\eta - erf\beta)}\right] T' = 0 \quad \ldots \quad (3·12)$$

The solution of this equation is

$$T = T_\infty - D_1 \int_\eta^\infty \frac{e^{-\eta^2}\, d\eta}{\left[\frac{2}{\sqrt{\pi}} + \epsilon\beta e^{\beta^2}(erf\eta - erf\beta)\right]^{2\sigma}} \quad \ldots \quad (3·13)$$

where σ denotes the Prandtl number of the fluid and D_1 is a constant to be determined from (2·8). Its value is given by

$$D_1 = \left[\frac{\lambda_2}{\lambda_1}\,\frac{2}{\sqrt{\pi}}\,\frac{T_S - T_A}{erf\beta\sqrt{\sigma\frac{a_1}{a_2}}}\sqrt{\sigma\frac{a_1}{a_2}}\; e^{-\beta^2\sigma\frac{a_1}{a_2}} - 2(1+\epsilon)\,\theta\,\beta\sigma\right] e^{\sigma\beta^2}\left(\frac{2}{\sqrt{\pi}}\right)^{2\sigma} \quad (3·14)$$

Substituting for D_1 from (3·14) in (3·13), we get

$$T = T_\infty + \left[2(1+\epsilon)\,\theta\,\beta\,\sigma - \frac{\lambda_2}{\lambda_1}\,\frac{2}{\sqrt{\pi}}\,\frac{T_S - T_A}{erf\beta\sqrt{\sigma\frac{a_1}{a_2}}}\sqrt{\sigma\frac{a_1}{a_2}}\; e^{-\beta^2\sigma\frac{a_1}{a_2}}\right] \times$$

$$e^{\sigma\beta^2}\left(\frac{2}{\sqrt{\pi}}\right)^{2\sigma} \int_\eta^\infty \frac{e^{-\eta^2}\, d\eta}{\left[\frac{2}{\sqrt{\pi}} + \epsilon\beta e^{\beta^2}(erf\eta - erf\beta)\right]^{2\sigma}}$$

$$\text{or} \quad \frac{T - T_\infty}{\theta} = \left[2(1+\epsilon)\,\beta\sigma - \frac{\lambda_2}{\lambda_1}\,\frac{2}{\sqrt{\pi}}\,\frac{T_S - T_A}{\theta}\,\frac{e^{-\beta^2\sigma\frac{a_1}{a_2}}}{erf\beta\sqrt{\sigma\frac{a_1}{a_2}}}\sqrt{\sigma\frac{a_1}{a_2}}\right] \times$$

$$e^{\sigma\beta^2}\left(\frac{2}{\sqrt{\pi}}\right)^{2\sigma}\int_{\eta}^{\infty}\frac{e^{-\eta^2}d\eta}{\left[\frac{2}{\sqrt{\pi}}+\epsilon\beta e^{\beta^2}(erf\eta - efβ)\right]^{2\sigma}} \quad \ldots \quad (3\cdot15)$$

Equation (3·15) gives the value of β by applying the other boundary condition of $T = T_S$ at $\eta = \beta$. The value of β depends upon the parameters $\frac{T_S - T_\infty}{\theta}$, $\frac{T_S - T_A}{\theta}$, λ_1, λ_2, ϵ and σ and a full discussion under such conditions is a one of considerable difficulty.

DISCUSSION OF THE RESULTS

Equation (3·15) has been discussed under two different situations; first by taking the value of σ to be unity and then at 7 implying thereby that the solid and the fluid have been taken to be ice and water respectively. By taking the value of σ to be unity, the integral in the equation (3·15) can be evaluated explicitly and we obtain

$$\frac{T - T_\infty}{\theta} = \left[2(1+\epsilon) - \frac{\lambda_2}{\lambda_1}\frac{2}{\sqrt{\pi}}\frac{T_S - T_A}{\theta}\frac{e^{-\beta^2\frac{a_1}{a_2}}}{\beta erf\beta\sqrt{\frac{a_1}{a_2}}}\sqrt{\frac{a_1}{a_2}}\right]\times$$

$$\frac{2}{\sqrt{\pi}}\frac{\beta e^{\beta^2}(1 - erf\eta)}{\left[\frac{2}{\sqrt{\pi}}+\epsilon\beta e^{\beta^2}(1 - erf\beta)\right]\left[\frac{2}{\sqrt{\pi}}+\epsilon\beta e^{\beta^2}(erf\eta - erf\beta)\right]} \quad (4\cdot1)$$

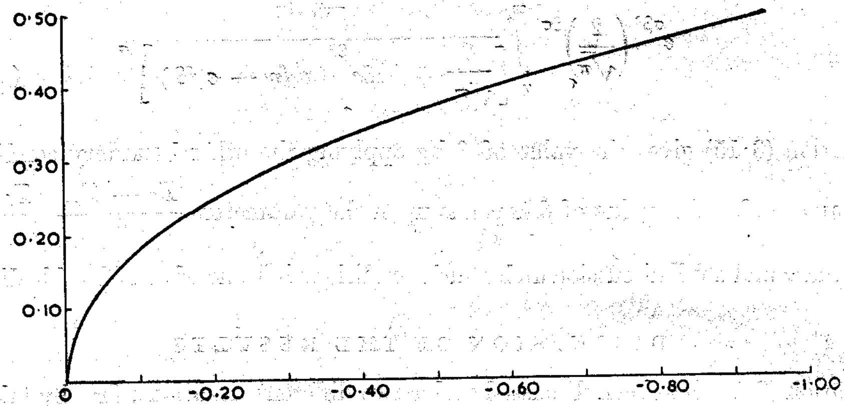
Now at $\eta = \beta$, $T = T_S$ and we get

$$\frac{T_S - T_\infty}{\theta} = \left[2(1+\epsilon) - \frac{\lambda_2}{\lambda_1}\frac{2}{\sqrt{\pi}}\frac{T_S - T_A}{\theta}\frac{e^{-\beta^2\frac{a_1}{a_2}}}{\beta erf\beta\sqrt{\frac{a_1}{a_2}}}\sqrt{\frac{a_1}{a_2}}\right]\times$$

$$\frac{\beta e^{\beta^2}(1 - erf\beta)}{\frac{2}{\sqrt{\pi}}+\epsilon\beta e^{\beta^2}(1 - erf\beta)} \quad \ldots \quad \ldots \quad (4\cdot2)$$

In equation (4·2) above if the surface temperature T_A and the interface temperature T_S are taken to be equal, the results of equations (27) of Chambre[1] follow.

For value of σ to be 7, the integral of the expression (3·15) is not directly integrable. Numerical integration is done to evaluate the integral and by taking the physical values of the parameters for ice and water, a graph is drawn for values of β against $\frac{T_A}{\theta}$



The growth rate β and its Dependance upon $\frac{T_A}{\theta}$

From the graph it will be clear that as $\frac{T_A}{\theta}$ decreases, more and more growth rate is obtained. Thus to have a growth rate of 0·20, the value of T_A should be $-10°C$ and to have a growth rate of 0·30, T_A should be $-24°C$.

Acknowledgement—The author is grateful to Dr. M. P. Murgai, for his keen interest and encouragement, and to the Research and Development Organisation for permission to publish this paper.

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