"POINTWISE AVAILABILITY OF A SYSTEM HAVING A GENERAL FAILURE TIME DISTRIBUTION"

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ABSTRACT

This paper considers the behaviour of a system having a general failure time distribution with probability density S (x). The Laplace transforms of various joint probabilities have been obtained and the corresponding results for some particular cases are deduced. In the end the behaviour of the system under steady state is examined.

INTRODUCTION

Hosford[1] considers the pointwise availability of a system with constant failure and repair time distributions. In this paper Hosford's assumptions are relaxed in as much as that the failure time distribution is assumed to have a probability density $S(x)$. The utilisation of such a density function brings forward the effect of ageing on the system. Further the employment of joint probabilities of the kind which includes the contingency that the system has suffered n failures by the time t is carried out with a view to extract more information about the behaviour of the system. It may be observed that the following probabilities *viz.*,

(*a*) Absolute probability that the system has suffered n failures by time t.

(*b*) Absolute probability that the system is operable at time t,

are easily obtainable from the corresponding joint probabilities defined in the body of the paper. The supplementary-variable[2] technique has been employed to obtain the solution.

DIFFERENTIAL EQUATIONS GOVERNING THE BEHAVIOUR OF THE SYSTEM

Define,

$P_{n,o}(x,t)\Delta x \equiv$ the joint probability that n failures have occurred by the time t, the system is operable at time t and the elapsed time since it was last commissioned for operation is in the interval $(x, x+\Delta x)$.

$P_{n,F}(t) \equiv$ the joint probability that the system has suffered n failures by time t and that it is in the outage stage at time t.

Elementary probability considerations lead to the following forward equations for the process:—

$$P_{n,F}(t+\Delta) = P_{n,F}(t)[1-\mu\Delta] + \int_0^\infty P_{n-1,0}(x,t)\eta(x)dx \cdot \Delta \quad (1)$$

$$n > 1$$

$$P_{1,F}(t+\Delta) = P_{1,F}(t)[1-\mu\Delta] \qquad \ldots \qquad (2)$$

$$P_{n,o}(x+\Delta, t+\Delta) = P_{n,o}(x,t)[1-\eta(x)\Delta] \quad n \geqslant 1 \qquad \ldots \qquad (3)$$

where $\eta(x)\,\Delta$ is the first order probability that the system fails in the time interval between x and $x+\Delta$, conditioned that the system has not failed upto time x since it was last commissioned for operation. The relation[3] between $\eta(x)$ and the probability density S(x) is given by,

$$S(x) = \eta(x) e^{-\int_{o}^{x} \eta(x)dx}$$

Equations (1) through (3) are to be solved under the boundary condition:

$$P_{n,o}(o,t) = \mu P_{n,F}(t) \qquad \ldots \qquad (4)$$

which arises from the fact that as soon as the system is repaired, it is commissioned for operation, and initially if we count time from an instant when the system has suffered one failure and happens to be in the failed state, then,

$$P_{n,F}(o) = 0, \quad n>1$$
$$= 1, \quad n=1$$

Equations (1) through (3) become when $\Delta \to 0$,

$$\frac{\partial P_{n,F}(t)}{\partial t} + \mu P_{n,F}(t) = \int_{o}^{\infty} P_{n-1,o}(x,t)\eta(x)dx \qquad \ldots \qquad (5)$$

$$\frac{\partial P_{1,F}(t)}{\partial t} + \mu P_{1,F}(t) = 0 \qquad \ldots \qquad (6)$$

$$\frac{\partial P_{n,o}(x,t)}{\partial t} + \frac{\partial P_{n,o}(x,t)}{\partial x} + \eta(x)P_{n,o}(x,t) = 0 \qquad \ldots \qquad (7)$$

Let the Laplace transform of the function $f(t)$ be denoted by

$$\bar{f}(s), \text{ i.e. } \bar{f}(s) = \int_{o}^{\infty} e^{-st} f(t)dt, \; Re\,(s) \geqslant 0$$

(see ref. 4).

Applying the Laplace transform, the equations (4) through (7) with the initial conditions mentioned above become,

$$\bar{P}_{n,o}(o,s) = \mu \bar{P}_{n,F}(s) \qquad \ldots \qquad (8)$$

$$(s+\mu)\bar{P}_{n,F}(s) = \int_{o}^{\infty} \bar{P}_{n-1,o}(x,s)\eta(x)dx \quad \ldots \quad (9)$$

$$(s+\mu)\,\bar{P}_{1,F}(s) = 1 \quad \ldots \quad (10)$$

$$\text{and } \frac{\partial \bar{P}_{n,o}(x,s)}{\partial x} + [s+\eta(x)]\,\bar{P}_{n,o}(x,s) = 0 \quad \ldots \quad (11)$$

which on solution gives,

$$\bar{P}_{n,o}(x,s) = \bar{P}_{n,o}(o,s)e^{-sx} \,.\, e^{-\int_{o}^{x}\eta(x)dx} \quad \ldots \quad (12)$$

With the help of equations (12) and (8), equation (9) gives,

$$(s+\mu)\bar{P}_{n,F}(s) = \mu\bar{P}_{n-1,F}(s).\ \ \bar{S}(s) \quad \ldots \quad (13)$$

where $\bar{S}(s)$ is the Laplace transform of S(x)

Also from (10), we have,

$$\bar{P}_{1,F}(s) = \frac{1}{s+\mu}$$

Therefore from (13), we get,

$$\bar{P}_{n,F}(s) = \left\{\frac{\mu^{n-1}}{(s+\mu)^n}\right\}\left[\bar{S}(s)\right]^{n-1} \quad \ldots \quad (14)$$

$$\text{and } \bar{P}_{n,o}(o,s) = \left\{\frac{\mu^n}{(s+\mu)^n}\right\}\left[\bar{S}(s)\right]^{n-1} \quad \ldots \quad (15)$$

Again,

$$\bar{P}_{n,o}(s) = \int_{o}^{\infty} \bar{P}_{n,o}(x,s)dx$$

$$i.e.\ \bar{P}_{n,o}(s) = \bar{P}_{n,o}(o,s)\left[\frac{1-\bar{S}(s)}{s}\right] \quad \ldots \quad (16)$$

where $\bar{P}_{n,o}(s)$ is the Laplace transform of the probability that $P_{n,o}(t)$ the system has suffered n failures by the time t and is in the operable state at time t.

Using (15) in (16), we get,

$$\bar{P}_{n,o}(s) = \left[\frac{\mu^n}{(s+\mu)^n}\right]\left[\bar{S}(s)\right]^{n-1}\left\{\frac{1-\bar{S}(s)}{s}\right\} \quad \ldots \quad \ldots \quad (17)$$

Employing the relation,

$$\bar{P}_n(s) = \bar{P}_{n,F}(s) + \bar{P}_{n,o}(s) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where $\bar{P}_n(s)$ is the Laplace transform of the probability that $P_n(t)$ the system has suffered n failures by time t which is equal to the sum of the Laplace transforms of the probabilities that the system is in the failure state with n failures and that the system is in operable state with n failures.

Substituting the values of $\bar{P}_{n,F}(s)$ and $\bar{P}_{n,o}(s)$ from (14) and (17) in (18), we get,

$$\bar{P}_n(s) = \left\{\frac{\mu^{n-1}}{(s+\mu)^n}\right\}\left[\bar{S}(s)\right]^{n-1}\left\{\frac{s+\mu-\mu\bar{S}(s)}{s}\right\} \quad \ldots \quad \ldots \quad (19)$$

Again

$$\bar{P}_F(s) = \sum_{n=1}^{\infty}\bar{P}_{n,F}(s) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where $\bar{P}_F(s)$ is the Laplace transform of $P_F(t)$, the probability that the system is in the failed state at time t.

Substituting the value of $\bar{P}_{n,F}(s)$ from (14) in (20) and summing, we have.

$$\bar{P}_F(s) = \frac{1}{s+\mu[1-\bar{S}(s)]} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Provided,

$$\left|\mu(\mu+s)^{-1}\bar{S}(s)\right| < 1$$

which gives the restriction that $R_e(s)$ should lie outside the roots of the equation.

$$x^2 - \mu x - \mu = 0$$

BEHAVIOUR OF THE SYSTEM UNDER STEADY STATE

To this end, we examine the limit of the expression, s[Laplace transform of $P_F(t)$] as $s \to o$ which gives P_F, the probability of the system being in the failed state when the state of statistical equilibrium has been reached.

From (21), we get,

$$s\overline{P}_F(s) = \frac{s}{s + \mu[1 - \overline{S}(s)]} \qquad \ldots \qquad (22)$$

whence $P_F = \left[1 + \frac{\mu}{l}\right]^{-1}$, where $\frac{1}{l}$ is the average life, under very weak conditions on S(x), *i.e.*, existence of the variance of life.

PARTICULAR CASES

(1) EXPONENTIAL FAILURE TIME DISTRIBUTION

Setting $\overline{S}(s) = \frac{\lambda}{s+\lambda}$ in equation (19), we get,

$$\overline{P}_n(s) = [\mu^{n-1} \lambda^{n-1}/(s+\mu)^n (s+\lambda)^n][s + \lambda + \mu] \qquad \ldots \qquad (23)$$

(2) K-ERLANG FAILURE TIME DISTRIBUTION

Setting $\overline{S}(s) = \left(\frac{\lambda}{s+\lambda}\right)^K$ in equation (19) we have

$$\overline{P}_n(s) = \left[\frac{\mu^{n-1} \lambda^{(n-1)K}}{s(s+\mu)^n (s+\lambda)^{nK}}\right]\left\{s(s+\lambda)^K + \mu(s+\lambda)^K - \mu\lambda^K\right\} \qquad \ldots \qquad (24)$$

Mean Number of Failures upto the time t

The mean number of failures upto the time t is given by the expression,

$$M(t) = \sum_{n=1}^{\infty} nP_n(t) \qquad \ldots \qquad (25)$$

Taking Laplace transform of the above expression, we have,

$$\overline{M}(s) = \sum_{n=1}^{\infty} n\overline{P}_n(s) \qquad \ldots \qquad (26)$$

Define the Generating function,

$$\overline{G}(\alpha, s) = \sum_{n=1}^{\infty} \overline{P}_n(s)\alpha^n \qquad \ldots \qquad (27)$$

Substituting the value of $\bar{P}_n(s)$ from relation (19) in relation (27) and summing the geometric progression under the condition $|\mu(\mu+s)^{-1}\bar{S}(s)|<1$, we get,

$$\bar{G}(\alpha, s) = \frac{\alpha[s+\mu-\mu\bar{S}(s)]}{s[s+\mu-\alpha\mu\bar{S}(s)]} \quad \ldots \quad (28)$$

Differentiating (28) with respect to α and then putting $\alpha = 1$, $\bar{M}(s)$, the Laplace transform of the mean number of failures upto the time t, is given by,

$$\bar{M}(s) = \frac{s+\mu}{s[s+\mu-\bar{S}(s)]} \quad \ldots \quad (29)$$

Relation (29) for a given probability density S (x) on inversion gives M(t). In the case of exponential failures time distribution,

$$\bar{S}(s) = \frac{\lambda}{s+\lambda}.$$

Substituting this value of $\bar{S}(s)$ in relation (29) we have,

$$\bar{M}(s) = \frac{(s+\mu)(s+\lambda)}{s^2(s+\lambda+\mu)} \quad \ldots \quad (30)$$

Relation (30) on inversion gives,

$$M(t) = \frac{\lambda^2(1+\mu t)+\mu^2(1+\lambda t)+\lambda\mu\{1+exp[-(\lambda+\mu)t]\}}{(\lambda+\mu)^2} \quad \ldots \quad (31)$$

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