STOCHASTIC LAW OF THE BUSY PERIOD FOR THE SIMPLE MACHINE INTERFERENCE PROBLEM

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ABSTRACT

In this note, the Laplace transform of the distribution of the length of the busy period for the simple machine interference problem has been obtained. It is shown that the mean length of the busy period under suitable limiting conditions corresponds to the mean length of the busy period for the infinite case.

INTRODUCTION

A busy period of a queuing process begins when a customer arrives to find the server free and starts getting service and ends when the system is again empty. The busy period is followed by an idle period and these two periods alternate. Obviously the distribution of the busy period is important from the point of view of server.

In this note, the distribution of the length of the busy period for the simple machine interference problem has been investigated. A repair man is to look after a set of N machines which fail from time to time. As soon as the machine fails the repair man starts repairing it and thus remains busy. In the mean time if more machines fail, they wait in a queue. The repairman becomes free once again only when all the machines are running. The distribution of the time intervals during which repairman is busy in repair activities is called the busy period distribution for the simple machine interference problem.

FORMULATION OF THE PROBLEM

Let us suppose that N machines are serviced by a single repair man. The machines are identical and work continuously. The running times of each machine are identically distributed positive random variables with the common distribution function.

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & if\ x \geqslant o \\ o & if\ \ x < o \end{cases} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

It is supposed that the repairman is idle if and only if there is no machine in the waiting line. In what follows the order of service is irrelevant. The times required for servicing the machines are independent positive random variables with a common probability density function $S(x)$. We define $\eta(x) \triangle$ as the conditional probability that a machine will be repaired between x and $x + \triangle$ subject to the condition that it was not repaired upto time x. It is easy to see that

$$S(x) = \eta(x) \exp\left\{ -\int_o^x \eta(y)\, dy \right\} \qquad (x \geqslant o) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Let the Laplace transform of $S(x)$ be $\bar{S}(s)$ so that

$$\bar{S}(s) = \int_{o}^{\infty} e^{-sx} S(x)\, dx$$

which is convergent whenever $Re(s) \geqslant o$

Let us assume that a unit arrives at time $t = 0$ which initiates the busy period and the system stops as soon as the empty state is reached at time t. If $P_o(t)$ denotes the probability that the system is empty at time t, then it is easy to see that $P_o(t)$ corresponds to the probability that the busy period is at most time t. Consequently the density function of the busy period distribution is given by

$$r(t) = \frac{d}{dt} P_o(t) \quad \ldots \quad (3)$$

Let us define $P_n(x, t)\, dx$ as the probability that at time t, there are n machines in the service facility and the machine under service has the elapsed service time which lies between x and $x + dx$

Simple continuity argument leads to the equations

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \left\{(N - n)\lambda + \eta(x)\right\}\right] P_n(x, t) = (N - n + 1)\lambda P_{n-1}(x, t) \quad (4)$$

$$(1 \leqslant n \leqslant N)$$

$$\frac{d}{dt} P_o(t) = \int_{o}^{\infty} P_1(x, t)\, \eta(x)\, dx \quad \ldots \quad (5)$$

which are to be solved subject to the following boundary conditions,

$$P_n(o, t) = \int_{o}^{\infty} P_{n+1}(x, t)\, \eta(x)\, dx \quad (1 \leqslant n < N) \quad \ldots \quad (6)$$

$$P_N(o,t) = o \quad \ldots \quad (7)$$

and the initial condition

$$P_1(x,o) = \delta(x) \quad \ldots \quad (8)$$

where $\delta(x)$ is the Dirac delta function

SOLUTION OF THE PROBLEM

We shall prove the following theorem for the distribution of the length of the busy period.

Theorem 1. If

$$\bar{r}(s) = \int_{o}^{\infty} e^{-st}\, r(t)\, (dt)$$

which is convergent for $Re(s) \geqslant o$, then,

$$\bar{r}(s) = \frac{\sum_{l=o}^{l=N-1} \binom{N-1}{l} \frac{1}{C_{l-1}(s)}}{\sum_{l=o}^{l=N} \binom{N}{l} \frac{1}{C_{l-1}(s)}} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where

$$C_r(s) = \prod_{m=o}^{m=r} \frac{\bar{S}[m\lambda + s]}{1 - \bar{S}[m\lambda + s]} \quad o \leqslant r \leqslant (N-1) \quad \ldots \quad \ldots \quad (10)$$

and $\quad C_{-1}(s) = 1$

Proof: Applying the Laplace transform in equation (4) with the initial condition (8) we get

$$\left[\frac{\partial}{\partial x} + \left\{ (N-n)\lambda + s + \eta(x) \right\}\right]\bar{P}_n(x,s) = (N-n+1)\lambda \bar{P}_{n-1}(x,s) \ldots \quad (11)$$

$$(1 < n \leqslant N)$$

$$\left[\frac{\partial}{\partial x} + \left\{ (N-1)\lambda + s + \eta(x) \right\}\right]\bar{P}_1(x,s) = \delta(x) \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

The equations (11)and (12) can be solved recurssively and the solution is given by

$$\bar{P}_n(x,s) = \left[\sum_{j=o}^{j=n-1} \left\{ \binom{N-n+j}{j} \bar{P}_{n-j}(o,s) \left(1 - e^{-\lambda x}\right)^j \right\} + \binom{N-1}{n-1}\left(1 - e^{-\lambda x}\right)^{n-1} \right]$$

$$\times \exp\left[-\left\{ (N-n)\lambda + s)x + \int_o^x \eta(y)\,dy \right\}\right] (1 \leqslant \eta \leqslant N) \quad \ldots \quad (13)$$

The boundary conditions (6) and (7) on taking Laplace transform, reduce to,

$$\bar{P}_n(o,s) = \int_o^\infty \bar{P}_{+1}(x,s)\,\eta(x)dx \quad (1 \leqslant n < N) \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$\bar{P}_N(o,s) = o \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Now, changing n to $n+1$ in (13), substituting in (14) and integrating we obtain

$$\bar{P}_n(o,s) = \sum_{j=o}^{j=n} \binom{N-n-1+j}{j} \bar{P}_{n+1-j}(o,s) \sum_{l=o}^{l=j} (-)^l \binom{j}{l} \bar{S}\,[(N-n-1+l)\lambda + s]$$

$$+ \binom{N-1}{n} \sum_{l=o}^{l=n} [(-)^l \binom{n}{l} \bar{S}\,[N-n-1+l)\lambda + s] \quad \ldots \quad \ldots \quad (16)$$

$$(1 \leqslant n < N)$$

Interchanging the order of summation in the first part of (16) we get,

$$\bar{P}_n(o,s) = \sum_{l=o}^{l=n} (-)^l \binom{N-n-1+l}{N-n-1+l} \bar{S}\,[(N-n-1+l)\lambda+s] \sum_{j=l}^{j=n} \binom{N-n-1+j}{N-n-1+j} \bar{P}_{n+1-j}(o,s)$$

$$+ \binom{N-1}{n} \sum_{l=o}^{l=n} (-)^l \binom{n}{l}\bar{S}[(N-n+1+l)\lambda+s] \quad (1 \leqslant n < N) \qquad (17)$$

After grouping the terms in (17), simple manipulation yields,

$$\bar{P}_n(o,s) = \sum_{l=n}^{l=o} (-)^l \binom{N-n-1+l}{N-n-1} A_{N-n-1+l}(s)\ \bar{S}[(N-n-1+l)\lambda+s] \qquad (18)$$

$$(1 \leqslant n < N)$$

where

$$A_r(s) = \sum_{j=r}^{j=N-1} \binom{j}{r} \bar{P}_{N-j}(o,s) + \binom{N-1}{r} \quad (o \leqslant r \leqslant N-1)$$

Now, changing n to $N-j$ in (18) multiplying both sides by $\binom{j}{n}$ summing for $j=r$ to $(N-1)$, and adding $(N-1)$ to both sides, we get

$$\left\{1-\bar{S}[r\lambda+s]\right\} A_r(s) = \bar{S}[(r-1)\lambda+s]A_{r-1}(s) + \binom{N-1}{r}$$

$$-A_{N-1}(s)\bar{S}[(N-1)\lambda+s]\binom{N-1}{r} \quad .. \qquad (19)$$

$$(1 \leqslant r < N)$$

Let us now define

$$C'_r(s) = \prod_{m=1}^{m=r} \frac{S[(m-1)\lambda+s]}{1-\bar{S}[m\lambda+s]} \quad .. \quad .. \quad .. \quad .. \qquad (20)$$

The relations connecting Cr given by (10) and Cr' given by (20) are

$$C'_r(s) = \frac{1-\bar{S}[s]}{1-\bar{S}[r\lambda+s]}\ C_{r-1}(s) = \frac{1-\bar{S}[s]}{\bar{S}[r\lambda+s]}\ C_r(s) \quad .. \qquad (21)$$

we also notice that $C'o = 1$.

Dividing (19) by $\left\{1-\bar{S}[r\lambda+s]\right\}$ $c'r(s)$ we get

$$\frac{A_r(s)}{C'_r(s)} = \frac{A_{r-1}(s)}{C'_{r-1}(s)} + \frac{1}{1-\bar{S}(s)} \binom{N-1}{r} \frac{1}{C_{r-1}(s)}$$

$$-\frac{A_{N-1}(s)\,S[(N-1)\lambda+s]}{1-\bar{S}(s)} \binom{N}{r} \frac{1}{C_{r-1}(s)} \quad .. \qquad (22)$$

changing r to $(r-1), (r-2), \ldots 1$ successively in (22) and adding we get

$$\frac{A_r(s)}{C_r(s)} = A_0(s) + \frac{1}{1-\bar{S}(s)} \sum_{l=1}^{l=r} \binom{N-1}{r} \frac{1}{C_{l-1}(s)} - \frac{A_{N-1}(s)\, S[(N-1)\lambda+s]}{1-\bar{S}[s]} \sum_{l=1}^{l=r} \binom{N}{l} \frac{1}{C_{l-1}(s)} \quad \ldots (23)$$

Now, summing all the equations in (18) and adding 1 to both sides the value of Ao (s) is given by

$$A_0(s) = \frac{1 - A_{N-1}\bar{S}[N-1)\lambda+s]}{1-\bar{S}[s]} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

Now substituting (24) in (23) and after little rearrangement

we obtain,

$$\frac{A_r(s)}{C'_r(s)} = \frac{1}{1-\bar{S}[s]} \sum_{l=0}^{l=\gamma} \binom{N-1}{l} \frac{1}{C_{l-1}(s)} - \frac{A_{N-1}(s)\bar{S}[(N-1)\lambda+s]}{1-\bar{S}[s]} \sum_{l=0}^{l=\gamma} \binom{N}{l} \frac{1}{C_{l-1}(s)} \quad (25)$$

where $C_{-1}(s) = 1$ as defined earlier.

Letting $r = (N-1)$, the value of $A_{N-1}(s)$ is given by

$$A_{N-1}(s) = \frac{\dfrac{C'_{N-1}(s)}{1-\bar{S}[s]} \displaystyle\sum_{l=0}^{l=N-1} \binom{N-1}{l} \frac{1}{C_{l-1}(s)}}{1 + C_{N-1}(s) \displaystyle\sum_{l=0}^{l=N-1} \binom{N}{l} \frac{1}{C_{l-1}(s)}} \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

Combining (3) and (5), taking Laplace transform and substituting the value of $P_1(x, s)$ from (13) one can easily obtain

$$\bar{r}(s) = \bar{S}[(N-1)\lambda+s]\, A_{N-1}(s) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Substituting the value of (26) in (27),

$$\bar{r}(s) = \frac{C_{N-1}(s) \displaystyle\sum_{l=0}^{l=N-1} \binom{N-1}{l} \frac{1}{C_{l-1}(s)}}{1 + C_{N-1}(s) \displaystyle\sum_{l=0}^{l=N+1} \binom{N}{l} \frac{1}{C_{l-1}(s)}} \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

which gives the result stated in (9).

This completes the proof of the theorem.

Particular Case :

Let us suppose that the repair time distribution is exponential with mean rate μ.

Then $\bar{S}(s) = \frac{\mu}{\mu + s}$ and $C_l\ (s) = \prod_{m=o}^{m=l} \left(\frac{\mu}{m\lambda + s} \right)$. The busy period distribution is given by

$$\bar{r}\ (s) = \frac{\sum_{l=o}^{l=N-1} \binom{N-1}{l} \frac{s\,[s+\lambda]\,[s+2\,\lambda]\ \ldots\ [s+(l-1)\,\lambda]}{\mu^{l-1}}}{\sum_{l=o}^{l=N} \binom{N}{l} \frac{s\,[s+\lambda]\,[s+2\,\lambda]\ \ldots\ [s+(l-1)\,\lambda]}{\mu^{l-1}}} \quad \ldots \quad \ldots \quad (29)$$

MEAN LENGTH OF BUSY PERIOD

For the mean-length of busy period, we have the following,

Theorem 2: If $\eta = \int_o^\infty x\, S\,(x)\, d\,x < \infty$ then the mean-length of the busy period is given by

$$\int_o^\infty t\, r\ (t)\, dt = \eta \sum_{l=1}^{l=N} \binom{N-1}{l-1} \prod_{m=1}^{m=l-1} \left(\frac{1-\bar{S}\ [m\,\lambda]}{\bar{S}\ [m\,\lambda]} \right) \quad \ldots \quad \ldots \quad (30)$$

Where the empty product is equal to 1.

Proof: Before proceeding to prove the theorem we notice that

$$\frac{1}{C_l\ (o)} = \begin{cases} o & o \leqslant l \leqslant N-1 \\ 1 & l = -1 \end{cases} \quad \ldots \quad \ldots \quad (31)$$

and

$$\frac{d}{ds}\ \frac{1}{C_l\ (s)}\ \Big|_{s=o} = \begin{cases} \eta \prod_{m=1}^{m=l} \frac{1-\bar{S}\ [m\,\lambda)}{\bar{S}\ [m\,\lambda]} & 1 \leqslant l \leqslant N-1 \\ \eta & l = 0 \\ 0 & l = -1 \end{cases} \quad (32)$$

Now the expression (9) can be written as

$$\left[\sum_{l=o}^{l=N} \binom{N}{l} \frac{1}{C_{l-1}\ (s)} \right] \bar{r}\,(s) = \sum_{l=o}^{l=N-1} \binom{N-1}{l} \frac{1}{C_{l-1}\ (s)} \quad \ldots \quad (33)$$

Differentiating both sides with respect to s and setting $s = 0$

we obtain after using (31) and (32)

$$\int_0^\infty t\, r(t)\, dt = \eta \left[\sum_{l=1}^{N} \binom{N}{l} \prod_{m=1}^{m=l-1} \left(\frac{1-\bar{S}[m\lambda]}{\bar{S}[m\lambda]} \right) - \sum_{l=1}^{l=N-1} \binom{N-1}{l} \prod_{m=1}^{m=l-1} \left(\frac{1-\bar{S}[m\lambda]}{\bar{S}[m\lambda]} \right) \right] \quad (34)$$

which reduces to (30) after suitable simplification.

This completes the proof of the theorem.

In the case of exponential repair time distribution, the mean-length of the busy period reduces to

$$\sum_{l=1}^{N} \frac{(N-1)!}{(N-l)!} \left(\frac{\lambda}{\mu} \right)^{l-1} = F\left(\frac{\lambda}{\mu}, N-1 \right) \quad \ldots \quad (35)$$

where

$$F(X,n) = 1 + nX + n(n-1)X^2 + \ldots \ldots \ldots + n!\, X^n$$

The function $F(X,n)$ can be easily computed from the equations (Refer 3)

$$F(X,1) = 1 + X \quad \text{and} \quad F(X,n) = n\, X\, F[X,(n-1)] + 1$$

and hence the mean length of the busy period can be computed for any value of N.

THE LIMITING CASE

It can be shown that the simple machine interference problem reduces to ordinary queing process with the infinite number of sources under suitable limiting conditions. It has not been possible to take the limit of the distribution of the busy period in general but the following theorem is proved for the limiting value of the mean length of the busy period.

Theorem 3 : If $\lambda \to 0$, $N \to \infty$ such that $N.\lambda = \lambda' < \infty$ then the limiting value of the mean length of the busy period is given by

$$\lim \int_0^\infty t\, r(t)\, dt = \begin{cases} \dfrac{\eta}{1-\lambda'\eta} & \text{if } \lambda'\eta < 1 \\ & \text{if } \lambda'\eta \geqslant 1 \end{cases} \quad \ldots \quad (36)$$

Proof : If all the moments of the distribution $s(x)$ exist, then $S(s)$ admits an expansion of the form

$$\bar{S}(s) = 1 - \frac{\eta}{1!} s + \frac{E(x^2)}{2!} s^2 - \ldots \ldots \ldots \ldots \quad (37)$$

where $E(x^r)$ denotes the expected value of x^r.

Consequently the expression (30) can be written as

$$\int_0^\infty t\, r(t)\, dt = \eta \sum_{l=1}^{l=N} \binom{N-1}{l-1} \prod_{m=1}^{m=l-1} \frac{\frac{\eta\,[m\lambda]}{1!} - \frac{E(x^2)}{2!} \ldots}{1 - \frac{\eta}{1!}(m\lambda) + \frac{E(x^2)}{2!}(m\lambda)^2 - \ldots} \quad \ldots \quad (38)$$

Now

$$\binom{N-1}{l-1} = \frac{N^{l-1}}{(l-1)!}\left[1 + o\left(\frac{1}{N}\right)\right]$$

Hence (38) can be written as

$$\int_0^\infty t\, r(t)\, dt = \eta \sum_{l=1}^{l=N} \frac{1}{l-1}\left[1 + o\left(\frac{1}{N}\right)\right] \prod_{m=1}^{m=l-1} \frac{\frac{m\lambda'\eta}{1!} - \frac{(m\lambda')^2}{N}\frac{E(x^2)}{2!} + \cdots}{1 - \frac{m\lambda'\eta}{N\,1!} + \frac{E(x^2)}{2!}\frac{(m\lambda')^2}{N^2}} \quad \ldots \quad (39)$$

where $N\lambda = \lambda'$

Taking limit as $N \to \infty$ we obtain

$$\lim \int_0^\infty t\, r(t)\, dt = \eta \sum_{l=1}^{\infty} (\lambda' \eta)^{l-1} \quad (40)$$

which is the result given in (34). This completes the proof of the theorem.

It may be pointed out that the equation (34) can be obtained either by differentiating the Kendall-Takács functional equation for the busy period distribution, namely

$$r(s) = \bar{S}\,[\lambda^1(1 - r(s)) + s]$$

or by following elementary probability arguments as pointed out in ref. 3, p. 59. But here we have obtained this relation in a different way.

DISCUSSION

We have obtained in this paper, the stochastic law of the busy period. The mean length of the busy period has been derived. The other moments can also be obtained by successive differentiation of (9). It may be remarked that queue-length distribution and waiting time distribution for the simple machine interference problem have been obtained by Tákacs [6] using an altogether different technique. Palm[5], Ashcroft[1], Benson & Cox [2] have also considered machine interference problem under various assumption on the repair time distributions and obtained queue-length distribution. (For a complete bibliography see references 4 and 6). We believe that our approach to this problem is new and can be easily applied for more complicated simple machine interference problem with two types of failure. (See Jaiswal & Thiruvengadam)[4].

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