DEPENDABILITY OF A COMPLEX SYSTEM WITH GENERAL REPAIR TIME DISTRIBUTIONS

R. C. Garg

Naval Research Group, Defence Science Laboratory, Delhi.

ABSTRACT

In this paper, a complex system consisting of n components with constant failure rates $\lambda_1, \lambda_2, \ldots \lambda_n$ and general repair time distributions with independent probability densities $S_1(x), S_2(x), \ldots, S_n(x)$ is examined with regard to Dependability. An attempt has been made to obtain the pointwise availability of the system which may be used to obtain the other two measures of Dependability, viz.; Reliability and Interval availability.

INTRODUCTION

Dependability is one of the most important measures of any system where failure is possible. This is defined as the probability that the system will be able to operate when needed. There are many different possible measures of dependability. Three of the most important measures which are listed by Hosford[1] are:

(1) Pointwise availability: the probability that the system will be able to operate within the tolerances at a given instant of time. This is sometimes referred to as operational readiness.

(2) Reliability: the probability that the system will be able to operate without a failure for a given interval of time. Failure is defined as the inability to operate within the tolerances.

(3) Interval Availability: An expected fraction of a given interval of time that the system will be able to operate within the tolerances. This is often referred to as efficiency.

Hosford considers a system with exponential failure and repair time distributions. In this paper, a complex system consisting of n components is examined with regard to dependability under the assumption of exponential failure time distributions and general repair time distributions with probability densities $S_1(x), S_2(x), \ldots\ldots, S_n(x)$ respectively. The pointwise availability for such a system is obtained. The supplementary—Variable[3] technique is employed for the solution.

DIFFERENTIAL EQUATIONS GOVERNING THE BEHAVIOUR OF THE COMPLEX SYSTEM

Define,

$P_i(x,t).\triangle$ = the probability that at time t, the system which has failed because of the failure of the ith component is under repair and the elapsed repair time lies in the interval $(x, x+\triangle)$.

$P_o(t)$ = the probability that the system is in the operable state at time t.

The forward equations for the process may be seen to be:—

$$P_i(x+\triangle, t+\triangle) = P_i(x,t)\,[1-\eta_i(x)\,\triangle] \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_\circ(t+\Delta)=P_\circ(t)\left[1-\sum_{i=1}^{n}\lambda_i\Delta\right]+\int_o^\infty\sum_{i=1}^{n}P_i\ (x,t)\ \eta_i\ (x)\ dx.\ \Delta \quad \text{.. (2)}$$

where $\eta_i\ (x)\ \Delta$ is the first order probability that the system is repaired in the time interval between x and $x+\Delta$, conditioned that the system has not been repaired up to time x. The relation between $\eta_i\ (x)$ and the probability density $S_i\ (x)$ is given by,

$$S_i\ (x)=\eta_i\ (x)\ e^{-\int_\circ^x \eta_i\ (x)\ dx} \quad \text{.. .. .. (3)}$$

a detailed description of relation (3) appears in Davis[3] in connection with failure time distribution.

Equations (1) and (2) are to be solved under the boundary condition:

$$P_i\ (o,t)=\lambda_i\ P_\circ(t) \quad \text{.. .. .. (4)}$$

which specifies that as soon as the system fails because of the failure of the ith component, it is put under repair.

Equations (1) and (2) become when $\Delta \longrightarrow 0$;

$$\frac{P_i\ (x,t)}{\partial t}+\frac{\partial P_i\ (x,t)}{\partial x}+\eta_i\ (x)\ P_i\ (x,t)=0 \quad \text{.. .. .. (5)}$$

$$\frac{\partial P_\circ(t)}{\partial t}+\lambda P_\circ(t)=\int_o^\infty\sum_{i=1}^{n}P_i\ (x,t)\ \eta_i\ (x)\ dx \quad \text{.. .. .. (6)}$$

where $\sum_{i=1}^{n}\lambda_i=\lambda$

Assume that the system is operating initially, so that $P_\circ(\mathrm{o})=1$. Let the Laplace transoform[4] of the function $f\ (t)$ be denoted by $\bar{f}\ (s)$ i.e.

$$\bar{f}(s)=\int_o^\infty e^{-st} f\ (t)\ dt,\quad Re\ (s)\geqslant 0$$

Applying the Laplace transform, the equations (5), (6) and (4) with the initial condition mentioned above become,

$$\frac{\partial \bar{P_i}\ (x,s)}{\partial x}+[\ s+\eta_i\ (x)\ ]\ \bar{P}_i\ (x,s)=0 \quad \text{.. .. .. .. (7)}$$

$$(s+\lambda)\ \bar{P}_\circ(s)-1=\int_o^\infty\ \sum_{i=1}^{n}P_i\ (x,s)\ \eta_i\ (x)\ dx \quad \text{.. .. .. (8)}$$

$$\bar{P}_i\ (\mathrm{o},s)=\lambda_i\ \bar{P}_\circ(s) \quad \text{.. .. .. (9)}$$

Relation (7) on integration and simplification gives,

$$\overline{P}_i(x,s) = \overline{P}_i(o,s)\, e^{-sx}\, e^{-\int_o^x \eta_i(x)\,dx} \qquad \ldots \quad (10)$$

Using (10) in (8), we get,

$$(s+\lambda)\,\overline{P}_o(s) - 1 = \int_o^\infty \sum_{i=1}^n \overline{P}_i(o,s)\, e^{-sx}\, e^{-\int_o^x \eta_i(x)\,dx}\, \eta_i(x)\,dx$$

$$\text{i.e. } (s+\lambda)\,\overline{P}_o(s) - 1 = \sum_{i=1}^n \overline{P}_i(o,s) \int_o^\infty e^{-sx} \cdot \eta_i(x)\, e^{-\int_o^x \eta_i(x)dx}\, dx$$

Employing the relation (3) in the above expression, we have,

$$(s+\lambda)\,\overline{P}_o(s) - 1 = \sum_{i=1}^n \overline{P}_i(o,s) \int_o^\infty e^{-sx} \cdot S_i(x)$$

$$\text{i.e. } (s+\lambda)\,\overline{P}_o(s) - 1 = \sum_{i=1}^n \overline{P}_i(o,s)\, \overline{S}_i(s) \qquad \ldots \quad (11)$$

where $\overline{S}_i(s)$ is the Laplace transform of $S_i(x)$.

Substituting the value of $\overline{P}_i(o,s)$ from relation (9) in relation (11), we get,

$$\left(s + \lambda - \sum_{i=1}^n \lambda_i\, \overline{S}_i(s)\right) \overline{P}_o(s) = 1$$

$$\text{i.e. } \overline{P}_o(s) = \frac{1}{s+\lambda - \sum_{i=1}^n \lambda_i\, \overline{S}_i(s)} \qquad \ldots \quad (12)$$

Also setting the value of $\overline{P}_o(s)$ from relation (12) in relation (19), we have,

$$\overline{P}_i(o,s) = \frac{\lambda_i}{s + \lambda - \sum_{i=1}^n \lambda_i\, \overline{S}_i(s)} \qquad \ldots \quad (13)$$

Again $\overline{P}_i\ (s)$, the Laplace transform of the probability $P_i\ (t)$ that the system is under repair due to the failure of the ith component, is given by,

$$\overline{P}_i\ (s) = \int_0^\infty \overline{P}_i\ (x, s)\, dx. \qquad \ldots \qquad \ldots \qquad \ldots \quad (14)$$

Using the relation (10) in (14), we get,

$$\overline{P}_i\ (s) = P_i\ (o,s) \int_0^\infty e^{-sx} \cdot e^{-\int_0^x \eta_i\ (x)\, dx}\ dx$$

which on integration by parts gives,

$$\overline{P}_i\ (s) = \overline{P}_i\ (o,s) \left[\frac{1 - \overline{S}^i\ (s)}{s} \right] \qquad \ldots \qquad \ldots \quad (15)$$

Substituting the value of $\overline{P}_i\ (o,s)$ from relation (9) in relation (15), we get,

$$\overline{P}_i\ (s) = \lambda_i\ \overline{P}_o\ (s) \left[\frac{1 - \overline{S_i}\ (s)}{s} \right] \qquad \ldots \qquad \ldots \quad (16)$$

Using relation (12) in (16), we have,

$$\overline{\ }_i\ (s) = \frac{\lambda_i}{s} \left[\frac{1 - \overline{S}_i\ (s)}{s + \lambda - \sum_{i=1}^{n} \lambda_i\ \overline{S}_i\ (s)} \right] \qquad \ldots \quad (17)$$

Therefore $\overline{P}_F\ (s)$, the Laplace transform of the probability $P_F\ (t)$ that the system is under repair due to the failure of any one of the n components, is given by,

$$\overline{P}_F\ (s) = \sum_{i=1}^{n} \overline{P}_i\ (s)$$

Using the relation (17) in the above expression, we get,

$$\overline{P}_F\ (s) = \sum_{i=1}^{n} \frac{\lambda_i}{s} \left[\frac{1 - \overline{S}_i\ (s)}{s + \lambda - \sum_{i=1}^{n} \lambda_i\ \overline{S}_i\ (s)} \right] \qquad \ldots \quad (18)$$

The probabilities that the system is in the operable state or in the failed state may be obtained on inverting (12) and (18) respectively for a specified value of the probability density, $S_i\ (x)$.

PARTICULAR CASES

(1) *Exponential Repair time distribution*

For this case, $\overline{S}_i\ (s) = \dfrac{\mu_i}{s+\mu_i}$, Substituting this value of $\widetilde{S_i}\ (s)$ in relations (12), (17) and (18), we have,

$$\overline{P}_o(s) = \left[\frac{1}{s + \sum_{i=1}^{n} \frac{\lambda_i \ s}{s+\mu_i}} \right] \quad \ldots \quad (19)$$

$$\overline{P}_i\ (s) = \left[\frac{\lambda_i}{s+\mu_i} \cdot \frac{1}{s + \sum_{i=1}^{n} \frac{\lambda_i \ s}{s+\mu_i}} \right] \quad \ldots \quad (20)$$

and $$\overline{P}_F\ (s) = \sum_{i=1}^{n} \left\{ \frac{\lambda_i}{(s+\mu_i)\left(s + \sum_{i=1}^{n} \frac{\lambda_i \ s}{s+\mu_i}\right)} \right\} \quad \ldots \quad (21)$$

(2) *One Component with K-Erlang repair time distribution*

If there is a single component with K-Erlang repair time distribution, equations (12) and (18) become,

$$P_o(s) = \frac{1}{s + \lambda - \lambda\,(\mu/s+\mu)^K} \quad \ldots \quad (22)$$

and $$\overline{P}_F\ (s) = \left(\frac{\lambda}{s+\mu}\right) \cdot \frac{1}{s + \lambda - \lambda\,(\mu/s+\mu)^K} \quad \ldots \quad (23)$$

Relations (22) and (23) may be inverted for any given set of values of λ, μ and K.

(3) *One Component with Constant repair rate*

Setting $K=1$ in relations (22) and (23), we get,

$$\overline{P}_o(s) = \frac{1}{s + \lambda - \mu\lambda/s+\mu} \quad \ldots \quad (24)$$

and $$\overline{P}_F\ (s) = \frac{\lambda}{s\,(s+\lambda+\mu)} \quad \ldots \quad (25)$$

Relations (24) and (25) on inversion give,

$$P_o\,(t) = \frac{\mu + \lambda \exp\left[-(\lambda+\mu)\,t\right]}{\lambda+\mu} \quad \ldots \quad (26)$$

and $$P_F\ (t) = \frac{\lambda - \lambda \exp\left[-(\lambda+\mu)\,t\right]}{\lambda+\mu} \quad \ldots \quad (27)$$

Relations (26) and (27) correspond with their counterparts as given by Hosford[1].

Acknowledgements—The author is most grateful to Dr. S. S. Srivastava, Director of Scientific Research (Navy), for his kind encouragements and useful suggestions and to Dr. C. Mohan for supervising the investigations.

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