ON PARTICULAR SOLUTIONS OF $\nabla^4 \Phi = 0$ AND $E^4 \Phi = 0$.

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ABSTRACT

THE correspondence between the particular solutions of the equations $\nabla^4 \phi = 0$ and $F^4 \phi = 0$ are pointed out. The solutions obtained already by Bhatnagar[1] are compared. An elementary discussion of the operational equation $[F_1 F_2 (L_1 + L_2)]^2 \phi = 0$ is presented. The operations $E_\nu^{\,2}$, $E_\nu^{\,4}$, $H_\nu^{\,2}$ and $H_\nu^{\,4}$ are introduced.

INTRODUCTION

The Laplacian operator is denoted by ∇^2 and can be identified with $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in the cartesian system (rectangular).

The operator E^2 stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega}\frac{\partial}{\partial \omega}$. Recently, some particular solutions were obtained for the equation $E^4 \phi = 0$ by Bhatnagar [1]. The correspondence between the particular solution of $\nabla^4 \phi = 0$ and $E^4 \phi = 0$ was not brought forth in the paper, or atleast was not pointed out and hence some results relating the particular solutions of the biharmonic equation and $E^4 \phi = 0$ are presented here.

Considering the operator, $E^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega}\frac{\partial}{\partial \omega}$, it can be seen that $E^2 f(x, \omega) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \omega^2} - \frac{1}{\omega}\frac{\partial f}{\partial \omega}$ can be transformed into a form involving the operator ∇^2 by the simple substitution :

(1) $f = \omega F;$

$$\frac{\partial f}{\partial \omega} = \omega \frac{\partial F}{\partial \omega} + F \quad \text{and}$$

$$\frac{\partial^2 f}{\partial \omega_2} = \omega \frac{\partial^2 F}{\partial \omega_2} + 2 \frac{\partial f}{\partial \omega} \quad \text{and hence}$$

(2)
$$E^2 f = \omega \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\omega}\frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} \right)$$

We define ∇^2 in $(x, \omega, \phi,)$ coordinate system and express ∇^2 as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} + \frac{1}{r}\frac{\partial}{\partial \omega} + \frac{1}{\omega^2}\frac{\partial^2}{\partial \phi^2} \quad \text{and}$$

$$\nabla^2 \left(F e^{i\phi} \right) = \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\omega^2} + \frac{1}{\omega}\frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} \right) e^{i\phi}$$

where F has been assumed to be independent of ϕ.

Recalling the operation of ∇^2in the (x, ω,ϕ) system (cylindrical polar coordinates) on functions independent of ϕ, we may write,

$$(3) \quad \nabla^2_{(x,\omega)} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\omega^2} + \frac{1}{\omega}\frac{\partial}{\partial \omega},$$

It is easily seen that,

$$(4) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega}\frac{\partial}{\partial \omega}\right)\left(\omega F e i\right)$$

$$= \omega\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega}\frac{\partial}{\partial \omega} - \frac{1}{\omega^2}\frac{\partial^2}{\partial \phi^2}\right)\left(Fe i\right)$$

and hence,

$$(5) \quad E^2 f = E^2(\omega F) = \omega\left(\nabla^2{}_{(a,w,)} - 1/\omega^2\right) F.$$

To obtain a particular solution of the equation $E^2 f{=}0$, one only needs a particular solution of $\nabla^2\phi{=}0$, where ∇^2 is the Laplacian operator in (x, ω, ϕ) system and the particular solution Φ depending on "ϕ" as $\omega Fe^{i\varphi}$; If such a solution is found, f can be written as ωF;

[Note: The operator E^2 is independent of ϕ]

$E^4 f{=}E^2 (E^2 f)$

writing $f{=} \omega Fe^{i\varphi}$

$E^4 (\omega Fe^{i\phi}) = e^{i\phi} E^2 [E^2(\omega F)]$

Also, $\omega\nabla^2 (Fe^{i\phi}) = E^2 (\omega Fe^{i\phi})$

consequently

$$E^4 (\omega Fe^{i\phi}) = E^2 . E^2(\omega Fe^{i\phi})$$

$$= E^2 . \omega(\nabla^2 x,\omega - 1/\omega^2). Fe^{i\phi}$$

$$= \omega(\nabla^2 x,\omega - 1/\omega^2)^2 Fe^{i\phi}$$

$$= \omega\nabla^4 Fe^{i\phi} \text{ and hence}$$

$\omega\nabla^4 (Fe^{i\phi}) = E^4 (fe^{i\phi})$ [$\nabla^2 \equiv$ Three dimensional Lap'acian operator]

To obtain a particular solution of $E^4\phi{=}0$, one can search for a particular solution of $\nabla^4\Phi{=}0$ of the form $\Phi{=} Fe^{i\varphi}$ (F independent of ϕ) and 'convert it' to a solution of $E^4\Phi{=}0$ by multiplying F by ω.

To summarise:

consider the operators

$$E^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} - \frac{1}{\omega} \frac{\partial}{\partial \omega} \right) (x,\omega)$$

and

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} + \frac{1}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) (x,\omega\ \phi)$$

Assume a solution of the form $\Phi = F(x,\omega)\, e^{i\varphi}$ to the equation $\nabla^2 \Phi = 0$. Then, a particular solution of $E^2\Phi = 0$ is ωF; Also if $F(x,\omega)\, e^{i\varphi} = \Phi$ is a solution of $\nabla^4 \Phi = 0$; $\omega F = \psi$ is a solution of $E^4 \psi = 0$.

ILLUSTRATIONS

Obviously, if Φ satisfies the equation $\nabla^2 \Phi = 0$ it is a solution of $\nabla^4 \Phi = 0$ too.

Assuming the ϕ dependence of $\Phi = Fe^{i\varphi}$ as that of $e^{i\varphi}$ we seek the solutions of

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial F}{\partial \omega} - \frac{F}{\omega^2} = 0$$

It is well known that

$$\begin{matrix}\cosh \\ \sinh\end{matrix}\ mx \left\{ AJ_1 (m\omega) + BY_1 (m\omega) \right\}$$

(for any $m \neq 0$)

are among these.

Hence,

(7) $$\phi = \omega \begin{matrix}\cosh \\ \sinh\end{matrix}\ mx \left\{ AJ_1 (m\omega) + BY_1 (m\omega) \right\}$$

are solutions of $E^2 \Phi = 0$ and hence of $E^4 \varphi = 0$. Also,

7(*a*) $$\phi = \omega \begin{matrix}\cos \\ \sin\end{matrix}\ mx \left\{ A^1 I_1 (m\omega) + B^1 K_1 (m\omega) \right\}$$

are solutions of $E^2 \Phi = 0$ and $E^2 \Phi = 0$.

If f is such that $E^2 \Phi = f$ where f is a solution of $E^2 f = 0$, Φ will be a solution of $E^4 \Phi = 0$; Thus we generate some more particular solutions for $E^4 \Phi = 0$.

Since the general solution of

$$\frac{d^2 y}{d\omega} + \frac{1}{\omega} \frac{dy}{d\omega} + \left(m^2 - \frac{1}{\omega^2} \right) y = A J_1 (m\omega) + B Y_1 (m\omega)$$
$$= V(m\omega), \text{ say,}$$

can be expressed as,

$$(8) \quad J_1(m\omega)\left[C_1 - \int_a^\omega \omega\, V(mx) Y_1(mx)\, dx\right] + Y_1(m\omega)\left[C_2 + \int_\beta^\omega \omega\, V(mx)\, J_1(mx)\, dx\right]$$

where C_1, C_2, a and β are arbitrary constants, some particular solutions of $E^4 \Phi = 0$ take the form as expressed by equation 2·18 in the paper by Bhatnagart[1].

Simple manipulations result in the new particular solutions given by equation 2·26 of reference 1. Since

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega}\frac{\partial}{\partial\omega} - \frac{1}{\omega^2}\right)(A\omega + B/\omega) \equiv 0$$

and

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}\right)(a + bx + cx^2 + dx^3) = 0$$

some more solutions of $\nabla^4 \Phi = 0$ can be seen to be of the form,

$(a + bx + cx^2 + dx^3)\ (A\omega + B/\omega)\, e^{i\theta}$ and hence

$$(9) \quad (a + bx + cx^2 + dx^3)\quad (A\omega^2 + B) = \Phi$$

is a solution of $E^4\varphi = 0$.

(a, b, c, d, A and B are arbitrary constants)

[cf: equation 2·30 of reference 1]

Also from the solution of

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega}\frac{\partial}{\partial\omega} - \frac{1}{\omega^2}\right) f = A\omega + B/\omega,$$

it is seen that

(10) $[C_1 + (C_2 + C_3 \log\omega)\,\omega^2 + C_4\omega^4]\,(a + bx)$ is a solution of $E^4 \Phi = 0$. (equation 2·9 of reference 1).

By similar arguments,

$$\omega \begin{matrix}\cos\\ \sin\end{matrix}\ \lambda x\ \begin{matrix}I_1\\ K_1\end{matrix}(\lambda\omega); \qquad \omega \begin{matrix}\cosh\\ \sinh\end{matrix}\ \lambda x\ \begin{matrix}J_1\\ Y_1\end{matrix}(\lambda\omega)$$

(11) and

$$x\omega \begin{matrix}\cos\\ \sin\end{matrix}\ \lambda x\ \begin{matrix}I_1\\ K_1\end{matrix}(\lambda\omega); \qquad x\omega \begin{matrix}\cosh\\ \sinh\end{matrix}\ \lambda x\ \begin{matrix}J_1\\ Y_1\end{matrix}(\lambda\omega)$$

can be proved to be particular solutions (cf: equation 2·39 reference 1)

Polar coordinates:

The correspondence, that has been proved to exist, between the solutions of $E^4\Phi = 0$ and $\nabla^4 \Phi = 0$ is particularly useful in helping us to choose the proper particular solutions in other systems of coordinates.

Considering, for instance, spherical polar coordinates for which $\nabla^2 \mho$ assumes the form

$$\nabla^2 \mho = \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left(r^2 \frac{\partial^2 \mho}{\partial r^2} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \mho}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \mho}{\partial \phi^2} \right\},$$

it may first be pointed out that only those solutions of $\nabla^2 \mho = 0$ *or* $\nabla^4 \mho = 0$ whose dependance on ϕ is as $e^{i\phi}$ are of interest in the discussion of $E^4 \Phi = 0$.

It may be recalled that a typical solution of $\nabla^2 \Phi = 0$ which is of the form $F(r, \theta)\ e^{i\phi}$ is given by

(12) $F = (Ar^n + Br^{-n-1}) \ [CP'_n(\mu) + DQ'_n(\mu)]$

The particular solutions of $\dfrac{1}{r^2} \dfrac{\partial}{\partial r} \left(r^2 \dfrac{\partial \mho}{\partial r} \right) = Ar^n + Br^{-n-1}$

will be of the form $A'r^{n+2} + B'r^{-(n-1)}$ and hence there exist particular solutions of $E^4 \Phi = 0$ which are of the form,

(13) $\left(\dfrac{B}{r^n} + A'r^{n+3} + \dfrac{B'}{r^{n-2}} + Ar^{n+1} \right) \left\{ a_1 P'_n(\mu) + b_1 Q'_n(\mu) \right\} \left(r\sqrt{1-\mu^2} \right).$

(cf: equation 3·19, reference 1)

In the degenerate case, n=0, we may easily prove that the solutions will be

(14) $(B + A'r^3 + B'r^2 + Ar)\ (a_1 + a_2\mu).$

It can be shown that a solution of $\nabla^4 \Phi = 0$

can be given in the form $r^m f(\mu)$ where f is a solution of

$$\left[\frac{\partial}{\partial \mu} \left(1 - \mu^2 \right) \frac{\partial}{\partial \mu} \right) + (m-2)(m-1) - \frac{1}{1-\mu^2} \right] \left\{ \frac{\partial}{\partial \mu} \left[(1-\mu^2) \frac{\partial}{\partial \mu} \right] + m(m+1) - \frac{1}{1-\mu^2} \right\} f = O$$

A typical solution will be given by

(15) $\left\{ \dfrac{\partial}{\partial \mu} \left((1-\mu^2) \dfrac{\partial}{\partial \mu} \right) + m(m+1) - \dfrac{1}{1-\mu^2} \right\} f = O$

which can be expressed as $A\, P'_m(\mu) + B\, Q'_m(\mu)$

Also, If

(16) $\left\{ \dfrac{\partial}{\partial \mu} \left((1-\mu^2) \dfrac{\partial}{\partial \mu} \right) + m(m+1) - \dfrac{1}{1-\mu^2} \right\} P(\mu) = A_1 P'_{m-2}(\mu) + B_1 Q'_{m-2}(\mu),$

(17) $\begin{cases} r^{m+1} f(\mu) \sqrt{1-\mu^2} & \text{is a typical solution of } E^4\phi = O \text{ with } f(\mu) \text{ of the form} \\ A\, P'_m(\mu) + B\, Q'_m(\mu) + A_1 P'_{m-2}(\mu) + B_1 Q'_{m-2}(\mu) \end{cases}$

If $m=1$, it can be easily shown that the solutions so derived will correspond to those obtained already in equation 3·30* of reference 1.

In the particular case of solving the equation $(L_1+L_2)^2 \phi=0$ where the linear differential operators L_1 and L_2 are independent—in the sense that L_1 operates only on $f(\xi_1)$ and L_2 on $\psi(\xi_2)$ where $f(\xi_1)$ and $\psi(\xi_2)$ are arbitrary functions of the coordinates ξ_1 and ξ_2. As an illustration, we may note Laplacian $\nabla^2_{x,\omega}$ in cylindrical coordinates, is of this form

$$\nabla^2_{(x,\omega)} \equiv L_1+L_2 ; \quad L_1 \equiv \frac{\partial^2}{\partial x^2} ;$$

$$L_2 \equiv \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega}\frac{\partial}{\partial \omega} ;$$

To obtain particular solutions of $(L_1 + L_2)^2 \phi = 0$ ie. $(L_1^2 + 2L_1L_2 + L_2^2) \phi = 0$, we assume ϕ to be of the form $X(\xi_1)\, Y(\xi_2)$ and dividing the equation by XY, one obtains $\frac{L_1^2 (X)}{X} + 2\frac{L_1 (X)}{X} \cdot \frac{L_2(Y)}{Y} + \frac{L_2^2(Y)}{Y} = 0.$ This suggests that the solution can be determined by solving for X and Y from

(18) $\quad (L_1 \pm \lambda) X = 0$ and $(L_2 \mp \lambda) Y = g(\xi_2)$ where $(L_2 \mp \lambda) g = 0$

or

(18a) $\quad (L_2 \pm \lambda) Y = 0$ and $(L_1 \mp \lambda) X = f(\xi_1)$ where $(L_1 \mp \lambda) f = 0$

where λ is a constant.

Illustrations:

$$L_1 = \frac{\partial^2}{\partial x^2}$$

$$L_2 = \frac{1}{\omega}\frac{\partial}{r}\left(\omega \frac{\partial}{\partial r}\right) - 1/\omega^2$$

If $(L_1 \pm \lambda^2) f = 0$,

$$f(\xi_1) = a \begin{matrix}\cos\\ \cosh\end{matrix} \lambda X + b \begin{matrix}\sin\\ \sinh\end{matrix} \lambda X$$

according as the upper or lower sign is chosen. Hence any solution of $(L_1 \mp \lambda) X = f(\xi_1)$ can be expressed as

$$X \equiv \begin{matrix}\cos\\ \cosh\end{matrix} \lambda x(a_1 + b_1 x) + \begin{matrix}\sin\\ \sinh\end{matrix} \lambda x(c_1 + d_1 x).$$

If $\lambda = 0$, it can be easily proved that $X = a_1 + b_1 x + c_1 x^2 + d_1 x^3$.

Similarly, if $(L_2 \pm \lambda) Y = 0$,

$$Y = A_1 \begin{matrix}I_1\\ J_1\end{matrix} (\lambda\omega) + B_1 \begin{matrix}K_1\\ Y_1\end{matrix} (\lambda\omega) \quad \text{and} \quad (A_1\omega + B_1/\omega) \quad (\lambda = 0)$$

*It may be pointed out here that the solutions $r\mu^2$, $r^2\mu^2$ are not "new" as given in reference 1, equation (3·30) but are implicitly stated through equations 3·19 and 3·8 of reference 1 for n=1. Also refer Appendix.

Hence the particular functions ωXY expressed as

$$I : \omega \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} \left\{ \frac{\cos}{\cosh} \lambda x \,(a_1 + b_1 x) + \frac{\sin}{\sinh} \lambda x \,(c_1 + d_1 x) \right\} \lambda \neq 0.$$

where A_1, B_1, a_1, b_1, c_1, d_1 and λ are arbitrary constants and

$$II : \omega \,(A_1\omega + B_1/\omega)\,(a_1 + b_1 x + c_1 x^2 + d_1 x^3)\ ,\ (\lambda = 0)$$

are solutions of $E^4\,(\omega XY) = 0$.

Also, the solution of

$$(L_2 \pm \lambda)\, Y = \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} \quad \lambda \neq 0$$

can be proved to be

$$C_1 \frac{I_1}{J_1} (\lambda\omega) + D_1 \frac{K_1}{Y_1} (\lambda\omega) - \left[\frac{I_1}{J_1} (\lambda\omega) \int \omega \frac{K_1}{Y_1} (\lambda\omega) \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} d\omega \right.$$
$$\left. - \frac{K_1}{Y_1} (\lambda\omega) \int \omega \frac{I_1}{J_1} (\lambda\omega) \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} d\omega \right]$$

where A_1, B_1, C_1 and D_1 are constants. Hence some solutions of $E^4(\Phi) = 0$ are

$$\Phi = \omega XY = \omega \left\{ a_1 \frac{\cos}{\cosh} \lambda x + b_1 \frac{\sin}{\sinh} \lambda x \right\} \delta \left\{ C_1 \frac{I_1}{J_1} (\lambda\omega) + D_1 \frac{K_1}{Y_1} (\lambda\omega) \right\}$$

III:
$$- \frac{I_1}{J_1} (\lambda\omega) \int \omega \frac{K_1}{Y_1} (\lambda\omega) \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} d\omega$$
$$- \frac{K_1}{Y_1} (\lambda\omega) \int \omega \frac{I_1}{J_1} (\lambda\omega) \left\{ A_1 \frac{I_1}{J_1} (\lambda\omega) + B_1 \frac{K_1}{Y_1} (\lambda\omega) \right\} d\omega$$

For $\lambda = 0$, ωXY will be of the form,

IV: $(a_1 + b_1 x)\, \omega \quad A_1\omega + B_1/\omega + C_1 \omega \log \omega + C_2 \omega^3$

The solutions I, II, III and IV have already been shown to be solutions of $E^4 \phi = 0$ in this paper, equations 7, 8, 9, 10 and elsewhere (reference 1, equations 2·9, 2·18, 2·26 2·31, 2·39 and 2·44)

In case the operator is of the form $f_1(\xi_1) f_2(\xi_2)(L_1 + L_2)$ where L_1 and L_2 are in terms of ξ_1 and ξ_2 coordinates only, let us assume a solution of the form $\dfrac{x(\xi_1)}{f_1(\xi_1)} \cdot \dfrac{y(\xi_2)}{f_2(\xi_2)}$

so that

$$f_1(\xi_1) f_2(\xi_2) \left[\frac{y}{f_2} L_1 \left(\frac{x}{f_1} \right) + \frac{x}{f_1} L_2 \left(\frac{y}{f_2} \right) \right] = f_1\, y\, L_1 \left(\frac{x}{f_1} \right) + x f_2 L_2 \left(\frac{y}{f_2} \right)$$

$$f_1 f_2 (L_1 + L_2) \left\{ \quad L_1 \left(\frac{x}{f_1} \right) + x f_2 L_2 \left(\frac{y}{f_2} \right) \right\} = f_1 f_2 \left\{ y\, L_1 \left[f_1 L_1 \left(\frac{x}{f_1} \right) \right] \right.$$
$$\left. + f_1 L_1 \left(\frac{x}{f_1} \right) L_2(y) + f_2 L_2 \left(\frac{y}{f_2} \right) L_1(x) + x L_2 \left(f_2 L_2 \left(\frac{y}{f_2} \right) \right) \right\}$$

and hence, representing the operator $f_1 f_2 (L_1 + L_2)$ by $f^{\delta}\xi$

$$\frac{f\delta^2\xi \left\{ \frac{xy}{f_1 f_2} \right\}}{xy} = 0 \text{ implies}$$

$$(19) \quad \frac{L_1\left[f_1 L_1\left(\frac{x}{f_1}\right)\right]}{x} + \frac{L_1(x)}{x}\frac{L_2(y)}{y} + \frac{L_2(y)}{y} \cdot \frac{f_1 L_1\left(\frac{x}{f_1}\right)}{x} + \frac{L_2[f_2 L_2(y/f_2)]}{y} = 0$$

For the special case when $f_2(\xi_2) = 1$

$$(20) \quad \frac{L_1\left\{f_1\left[L_1\left(\frac{x}{f_1}\right)\right\}\right]}{x} + \frac{L_2(y)}{y}\left\{\frac{L_1(x)}{x} + \frac{L_1(x/f_1)}{x/f_1}\right\} + \frac{L_2{}^2(y)}{y} = 0$$

It can be seen that the condition $\frac{L_2(y)}{y} = \lambda$ implies,

$$(21) \quad \frac{L_1\left\{f_1\left[L_1\left(\frac{x}{f_1}\right)\right]\right\}}{x} + \lambda \frac{L_1(x)}{x} + \frac{L_1(x/f_1)}{x/f_1}\Big\} + \lambda^2 = 0$$

from which an expression for X can be derived. If $\frac{L_1(x)}{x} = \alpha_1$ and also $\frac{L_1(x/f_1)}{x/f_1} = \alpha_2$ (i.e. if both X and X/f_1 are eigen functions of the operator L_1), we find that

$$\alpha_1 \alpha_2 + \frac{L_2(y)}{y}\left\{\alpha_1 + \alpha_2\right\} + \frac{L_2{}^2(y)}{y} = 0$$

(22) or $(L_2 + \alpha_1)(L_2 + \alpha_2)\, y = 0$

where α_1, α_2 are constants. This equation can be solved easily for Y thereafter, and the complete solution thus obtained.

ILLUSTRATION

Spherical coordinates:

$$L_1 \equiv \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \quad ; \quad f_1 = 1/r^2$$

$$\equiv \theta(\theta + 1) \quad ; \quad \theta = \frac{\partial}{\partial z} \text{ where } z = \log r.$$

For the operator L_1, r^n will be an eigen function with the eigen value $(n^2 + n)$ so that r^{n-2} will be an eigen function with the eigen value $(n^2 - 3n + 2)$.

i.e. $\frac{L_1(r^n)}{r^n} = n(n+1)$ and $\frac{L_1(r^n/r^2)}{r^{n-2}} = n^2 - 3n + 2$;

correspondingly, one can solve (for Y) the equation

$$[L_2 + n(n+1)]\,[L_2 + (n-2)(n-1)]\, y = 0$$

Some of the solutions can be,

$$Y = A_1 P'_n(\mu) + B_1 Q'_n(\mu) \text{ and}$$
$$A_1 P'_{n-2}(\mu) + B_1 Q'_{n-2}.$$

In particular, for $n = 2$

$Y = A_1 P_2'(\mu) + B_1 Q'_n(\mu)$ and

$(A_1' + B_1'\mu)/\sqrt{1-\mu^2}$ or $A_1 \mu\sqrt{1-\mu^2}$ and $(A_1' + B_1'\mu)/\sqrt{1-\mu^2}$,

confining to the Legendre function of the first kind. A solution can therefore be written as $A_1 \mu\sqrt{1-\mu^2}$ or $(A_1' + B_1'\mu)/\sqrt{1-\mu^2}$. Consequently, the solutions for $E^4\phi = 0$ can, hence be expressed as $rA_1\mu(1-\mu^2)$ and $r(A_1' + B_1'\mu)$

A_1, A_1' and B_1' : arbitrary constants).

Similarly one can derive the solutions corresponding to other powers.

Discussions of a similar nature can be extended to some other operators as well. Defining $E_\nu^2 \equiv \frac{\partial^2}{\partial\omega^2} - \frac{\nu}{\omega}\frac{\partial}{\partial\omega} + \frac{\partial^2}{\partial x^2}$, it can be proved easily that

$$(23)\quad E_\nu^2\left(\omega^{\frac{\nu+1}{2}} F\right) = \omega^{\frac{\nu+1}{2}}\left\{\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega}\frac{\partial}{\partial\omega} - \frac{(\nu+1)^2}{4\omega^2} + \frac{\partial^2}{\partial x^2}\right\}$$

which corresponds to the operation of the three dimensional Laplacian ∇^2 on a function which varies with ϕ as $e^{i\left(\frac{\nu+1}{2}\phi\right)}$

For $\nu = 1$, $E_\nu^2 \geqslant E^2$ discussed in the paper, and $\frac{\nu+1}{2} = 1$; Hence,

$$(24)\quad E_\nu^2\left(\omega^{\frac{\nu+1}{2}} F(\omega,x)\, e^{i\frac{\nu+1}{2}\phi}\right) = \omega^{\frac{\nu+1}{2}}\nabla^2\left(F e^{i\frac{\nu+1}{2}\phi}\right)$$

$$(24)\text{a}\quad E^{\nu 4}\left[\omega^{\frac{\nu+1}{2}} F(\omega, x)\, e^{i\frac{\nu+1}{2}\phi}\right] = \omega^{\frac{\nu+1}{2}}\nabla^4\left(F e^{i\frac{\nu+1}{2}\phi}\right)$$

and hence corresponding to a solution of $\nabla^4\left(F e^{i\frac{\nu+1}{2}\phi}\right) = 0$ with F independent f ϕ, a solution of $E_\nu^4 f = 0$ exists such that $f = \omega F^{\frac{\nu+1}{2}}$. An operator of the form $H_\nu \equiv \left(\frac{\partial^2}{\partial\omega^2} - \frac{\nu}{\omega}\frac{\partial}{\partial\omega} + \frac{\partial^2}{\partial x^2} - \kappa^2\right)$ will correspond to the Hamiltonian

$$(25)\quad (\nabla^2 - \kappa^2) \text{ since } H_\nu\left(\omega^{\frac{\nu+1}{2}} F\right) = \omega^{\frac{\nu+1}{2}}(\nabla^2 - \kappa^2)\left(F e^{i\frac{\nu+1}{2}\phi}\right)$$

with F independent of ϕ.

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APPENDIX

To solve

$$A(1)\quad \left(1-\mu^2\right)\frac{d^2}{d\mu^2}\left\{\left(1-\mu^2\right)\mu''\right\}+2a\left(1-\mu^2\right)\mu''+\left(a-3\right)\left(a-1\right)\mu=0$$

(*cf*: equation 3·22, Reference 1)

Denote by θ_1 and θ_2 the operators $\left(1-\mu^2\right)\frac{d^2}{d\mu^2}$ and

$$\left\{\left(1-\mu^2\right)\frac{d^2}{d\mu^2}-2\mu\frac{d}{d\mu}-\frac{1}{1-\mu^2}\right\}$$

respectively; By simple calculations, it may be shown that

$$\theta_1\ \{N(\mu)\sqrt{1-\mu^2}\} = \sqrt{1-\mu^2}.\ \theta_2\ \{N(\mu)\}$$

so that

$$\theta_1^2\ N(\mu)\sqrt{1-\mu^2} = \theta_1[\sqrt{1-\mu^2}.\ \ \theta_2^2\ \ N(\mu)\ ] = \sqrt{1-\mu^2}.\ \ \theta_2^2\, N(\mu)$$

Equation A(1) can be transformed into the form

$$A(2)\quad \theta_1^2\mu+2a\,\theta_1\mu+(a-3)\,(a-1)\,\mu=0.$$

$N\sqrt{1-\mu^2}$ is a solution of A(2), provided N satisfies the equation,

$$A(3)\quad [\theta_2^2+2a\theta_2+(a-3)\,(a-1)]\,N=0.$$

Denoting the roots $(-a\pm\sqrt{4a-3})$ of the equation $x^2+2ax+(a-3)(a-1)=0$ by a_1 and a_2 respectively, we find that

$$N = A_1\,P^1{}_{n-1}\,(\mu)+B_1\,Q'{}_{n-1}\,(\mu)$$
$$+\,A_2\,P^1{}_{n-3}\,(\mu)+B_2\,Q'{}_{n-3}\,(\mu)$$

where $3+n(n-3)$ has been written for a.

The solution of $(D^2-3D+3)\,R=aR$

(*cf*: equation 3·20, reference 1)

can be expressed as $R=C_1\,r^n+C_2r^{-(n-3)}$.

(note: $a=3+n(n-3)$). Hence, a solution of $E^4\,\Phi=0$ will be,

$$[C_1r^n+C_2r^{-(n-3)}][A_1\,P'{}_{n-1}\,(\mu)+B_1\,Q'{}_{n-1}\,(\mu)+A_2\,P'{}_{n-3}\,(\mu)+B_2\,Q^1{}_{n-3}\,(\mu)]\,(\sqrt{1-\mu^2})$$

(A_1, B_1, C_1, A_2, B_2, and C_2 are arbitrary constants).

This result has been implicitly stated through equation (17) of this paper.

In particular, for $n=1$, $a=1$ and the solutions given by equation 3·30 of reference (1) can be *deduced*.