EFFECT OF ROTATION AND A MAGNETIC FIELD ON RAYLEIGH-TAYLOR STABILITY

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ABSTRACT

In this paper the hydromagnetic stability of an incompressible fluid of variable density in the presence of magnetic field as well as rotation is considered. The magnetic field is taken along axis of rotation and stratified in the direction. A variational principle for the problem under consideration has been established. Two density configurations *viz.* (*i*) density varies continuously with height and (*ii*) two superposed fluids of great depths have been studied in detail.

INTRODUCTION

A NUMBER of papers on Rayleigh-Taylor Instability in the presence of magnetic field have appeared during past few years. They have been discussed by Chandrasekhar[1]. Nearly all of them referred to homogeneous magnetic fields taken in different directions. Talwar[2] has studied this problem for variable magnetic fields. The purpose of the present paper is to bring out the new facts when a Coriolis Force is also acting. The problem has been studied in 4 parts. In the first section the basic equations governing the problem at hand are derived. Then a variation principle is established and further it is shown that if coriolis force and magnetic field both act in the horizontal direction the validity of the principle imposes a condition which separates the hydrodynamics from electro-magnetism. In the 3rd part the stability of a continuously stratified fluid has been discussed. The last section deals with the stability of two superposed fluids. In this connection it may be pointed out that remarks made by Reid[3] do not any way point to reduce the effctiveness of the variational procedure in the present case. Also, in order to avoid the mathematical complications the fluid is assumed to be inviscid, incompressible and of infinite conductivity.

BASIC EQUATIONS

Consider a layer of conducting fluid confined between two horizontal planes. This is subjected to a magnetic field $[H_o(z),o,o]$ and rotation $\Omega\, \vec{\nu}$. The equations of the problem are, then

$$\rho\left[\frac{\partial u_i}{\partial t}+\frac{\partial\, u_i\, u_j}{\partial x_j}\right]=-\frac{\partial}{\partial x_i}\left(P+\tfrac{1}{2}\,|\,\Omega\times r\,|^2\right)-g\,\rho\,\lambda_i+2\,\rho\,\Omega\epsilon_{ijk}\,u_j\,\nu_k + \frac{\mu}{4\pi}\left[\,curl\ \vec{H}\times\vec{H}\,\right]_i \quad \ldots \quad \ldots \quad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \ldots \quad (2)$$

$$\frac{\partial \rho}{\partial t} + v_j\,\frac{\partial \rho}{\partial x_j} = 0 \quad \ldots \quad (3)$$

$$\frac{\partial \vec{H}}{\partial t} = \text{curl}\,(\vec{u}\times\vec{H}) \quad \ldots \quad (4)$$

and $$\text{div}\,\vec{H} = 0 \quad \ldots \quad (5)$$

where the symbols have their usual meaning.

The equilibrium situation is a static one and is characterized by $u_i = 0$. Initially

$$\left.\begin{aligned} \rho &= \rho_\circ(z) \\ p &= p_\circ(z) \\ H &= H_\circ(z) \end{aligned}\right\} \qquad \ldots \ldots \qquad (6)$$

Consider the static situation to be disturbed, so that

$$\left.\begin{aligned} &\rho = \rho_\circ + \delta\rho\,(x,y,z,t,); \quad p = p_\circ + \delta p\,(x,y,z,t,) \\ &H_i = H_\circ a_i + h_i\,(x,y,z,t) \text{ and } u_i = (u,v,w) \end{aligned}\right] \qquad \ldots \ldots \qquad (7)$$

where a_i is a unit vector in the direction of x-axis. The perturbations are assumed to be small and thus the second order quantities are neglected. We further assume that the fluid is rotating about z-axis (*i.e.* ν_k = o,o,1). The case when ν_k = (1,o,o) is also dealt with in the next section. The linearized form of equations (1)—(5) in the cartesian co-ordinate system are

$$\left.\begin{aligned} &\rho_\circ \frac{\partial u}{\partial t} - 2\rho_\circ \Omega v = -\frac{\partial}{\partial x}\delta p + \frac{\mu}{4\pi} h_z \frac{dH_\circ}{dz} \qquad (i) \\ &\rho_\circ \frac{\partial v}{\partial t} + 2\rho_\circ \Omega u = -\frac{\partial}{\partial x}\delta p + \frac{\mu}{4\pi} H_\circ\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right) \qquad (ii) \\ &\rho_\circ \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\delta p - g\delta\rho - \frac{\mu}{4\pi}\left[h_x \frac{dH_\circ}{dz} + H_\circ\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right)\right] \qquad (iii) \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (i) \\ &\frac{\partial}{\partial t}\delta\rho + w\frac{d\rho_\circ}{dz} = 0 \qquad (ii) \end{aligned}\right\} \qquad \ldots \ldots \qquad (9)$$

$$\left.\begin{aligned} &\frac{\partial h_x}{\partial t} + w\frac{dH_\circ}{dz} = H_\circ \frac{\partial u}{\partial x} \qquad (i) \\ &\frac{\partial h_y}{\partial t} = H_\circ \frac{\partial v}{\partial x} \qquad (ii) \\ &\frac{\partial h_z}{\partial t} = H_\circ \frac{\partial w}{\partial x} \qquad (iii) \\ &\text{and } \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0 \qquad (iv) \end{aligned}\right\} \qquad \ldots \ldots \qquad (10)$$

Following the usual practice in problems of this kind, we seek the solutions of these equations which are of the form

$$u, v, w, \delta\rho, \delta p, h = (\text{Some function of } z) \times e^{(ik_x x + ik_y y + nt)}$$

where k_x, k_y are horizontal wave numbers of the perturbation. Equations (8)—(10) then become

$$\left.\begin{aligned} &\rho_\circ n u - 2\,\Omega\,\rho_\circ v = -ik_x\,\delta p + \frac{\mu}{4\pi} h_z\, DH_\circ \quad (i) \\ &\rho_\circ n v + 2\,\Omega\,\rho_\circ u = -ik_y\,\delta p + \frac{\mu}{4\pi} H_\circ\,(i\,k_x\,h_y - ik_y\,h_x) \quad (ii) \\ &\rho_\circ n w = -D\delta p - g\delta\rho + \frac{\mu}{4\pi}\left[H_\circ\,(i\,k_x\,h_z - D\,h_x) - h_x DH_\circ\right] \quad (iii) \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} &ik_x\,u + ik_y\,v = -Dw \\ &n\,\delta\,\rho + w\,D\,\rho_\circ = 0 \end{aligned}\right\} \quad \ldots \quad (12)$$

$$\left.\begin{aligned} &nh_x + w\,D\,H_\circ = H_\circ ik_x\,u \quad (i)\,; \quad nh_y = H_\circ ik_x\,v \quad (ii)\,; \\ &nh_z = H_\circ\,ik_x\,w \quad (iii)\,; \quad ik_n\,h_x + ik_y\,h_y + D\,h_z = 0 \quad (iv) \end{aligned}\right\} \quad (13)$$

where $D = d/dz$

Let $\vec{\omega}$ be the vorticity vector and $\zeta(z)$ denote its z-components. So that

$$\zeta\,(z) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = i\,k_n\,v - i\,k_y\,u \quad \ldots \quad (14)$$

From equation (8)

$$ik_x\,h_y - ik_y\,h_x = \frac{1}{n}\left[H_\circ\,ik_x\,\zeta + ik_y\,w\,D\,H_\circ\right] \quad \ldots \quad (15)$$

Eliminating u and v from equations (11-*i*, *ii*) with the help of equations (12-*i*), (14) and (15), we obtain.

$$-\rho_\circ\,n\,D\,w - 2\,\Omega\,\rho_\circ\,\zeta = \delta\,p\,k^2 + \frac{\mu}{4\pi n}\left[-k^2\,w\,H_\circ\,D\,H_\circ - k_x\,k_y\,H_\circ^2\,\zeta\right] \quad \ldots \quad (16)$$

where $k^2 = k^2_x + k^2_y$

also from equation (11-*iii*) we can write after simple Algebra

$$\begin{aligned} k^2\,[D\,(\delta p) + g\,\delta\,\rho + \rho_\circ\,n\,w] = \frac{\mu}{4\pi n}\,[\,&2\,k^2_x\,H_\circ\,DH_\circ\,Dw + k^2_x\,H^2_\circ\,D^2 w \\ &+ 2\,k_x\,k_y\,H_\circ\,\zeta\,DH_\circ + k_x\,k_y\,H^2_\circ\,D\,\zeta - k^2\,k^2_x\,H^2_\circ\,w \\ &+ k^2 H_\circ DH_\circ D w + k^2 w\,H_\circ\,D^2 H_\circ + k^2 w\,(DH_\circ)^2\,] \quad \ldots \quad (17) \end{aligned}$$

operating with D on both sides of (16), substracting from (17) and utilizing (12-*ii*), we get

$$\begin{aligned} &k^2\left(\rho_\circ n w - \frac{g}{n}\,D\rho_\circ\,w\right) - n\,D\,(\rho_\circ\,Dw) - 2\,\Omega\,D(\rho_\circ\,\zeta) \\ &= \frac{\mu k^2_x}{4\pi n}\,[\,DH^2_\circ\,Dw + H^2_\circ\,(D^2 - k^2)\,w\,] \quad \ldots \quad (18) \end{aligned}$$

Also eliminating δp from (11-*i*), and (11-*ii*) we get

$$\rho_\circ n\,\zeta - 2\,\Omega\,\rho_\circ\,Dw = -\frac{\mu}{4\pi n}\,k^2_x\,H^2_\circ\,\zeta \quad \ldots \quad (19)$$

Boundary Conditions

The normal component of $\vec{u}$ (*i.e.* w) must vanish on the bounding surfaces. Thus

$$\left.\begin{array}{lll} (i) & w = 0 & \text{for } z = 0, d \\ (ii) & \zeta = Dw = 0 & \text{on a rigid surface} \\ (iii) & D\zeta = D^2 w = 0 & \text{on a free surface} \end{array}\right\} \quad \ldots \quad \ldots \quad (20)$$

Equations (18), (19) together with the boundary conditions (20) constitute the basic equations of the problem at hand. In the next section we shall establish a variational principle.

A VARIATIONAL PRINCIPLE

Let there be solutions w_i, w_j which, respectively correspond to the two characteristic values n_i and n_j of equations (18) and (19). ζ_i and ζ_j are also associated respectively with n_i and n_j. Multiply equation (18) by w_j (corresponding to n_j) and integrating over the range $0 \leqslant z \leqslant d$, we get

$$n_i^2 \left[k^2 \int_0^d \rho_0 w_i w_j - \int_0^d w_j D(\rho_0 D w_i) \right] - n_i \left[2\,\Omega \int_0^d w_j D(\rho_0 \zeta_i) \right]$$

$$= g k^2 \int_0^d D\rho_0 w_i w_j + \frac{\mu k_x^2}{4\pi} \left[\int_0^d (DH_0^2 w_j D w_i + H_0^2 w_j D^2 w_i) - k^2 \int_0^d H_0^2 \, w_i w_j \quad (21)\right.$$

Integration by parts leads to

$$n_i^2 \int_0^d \left[\rho_0 (Dw_i Dw_j + k^2 w_i w_j) \right] + \frac{\mu k_x^2}{4\pi} \int_0^d \left[H_0^2 (Dw_i\, Dw_j + k^2 w_i w_j) \right]$$

$$+ 2\Omega n_i \int_0^d \rho_0 \zeta_i \; Dw_j - gk^2 \int_0^d D\rho_0 w_i w_j = 0 \quad \ldots \quad \ldots \quad (22)$$

Multiply equation (19) for n_j by ζ_i and integrate by parts, we have

$$n_j \int_0^d \rho_0 \zeta_i \; \zeta_j - 2\Omega n_j \int_0^d \rho_0 \zeta_i \; Dw_j + \frac{\mu k_x^2}{4\pi} \int_0^d H_0^2 \, \zeta_i \; \zeta_j = 0 \quad \ldots \quad (23)$$

From (22) and (23), we obtain

$$n_i^2 \left[\int_0^d \rho_0 \,(Dw_i\, Dw_j + k^2\, w_i w_j) \right] + \frac{\mu k_x^2}{4\pi} \left[\int_0^d H_0^2\, (Dw_i\, Dw_j + k^2\, w_i\, w_j) \right]$$

$$+ n_i\, n_j \int_0^d \rho_0\, \zeta_i\, \zeta_j + \frac{n_i}{n_j}\, \frac{\mu k_x^2}{4\pi} \left[\int_0^d H_0^2\, \zeta_i\, \zeta_j - gk^2 \int_0^d D\rho_0\, w_i w_j = 0 \right. \quad (24)$$

Now put $i=j$, so that $n_i=n_j=n$ and etc. Then (24) becomes

$$n^2 \int_0^d \rho_0 \,[\,(Dw)^2 + k^2 w^2] + \frac{\mu k_x^2}{4\pi} \int_0^d H_0^2\, [\,(Dw)^2 + k^2 w^2] + n^2 \int_0^d \rho_0 \zeta^2$$

$$+ \frac{\mu k_x^2}{4\pi} \int_0^d H_0 \zeta^2 - gk^2 \int_0^d D\rho_0\, w^2 = 0 \quad \ldots \quad \ldots \quad (25)$$

$$\text{or} \quad n^2 I_1 + \frac{\mu k_x^2}{4\pi}\, I_2 + n^2 I_3 + \frac{\mu k_x^2}{4\pi}\, I_4 - gk^2\, I_5 = 0 \quad \ldots \quad \ldots \quad (26)$$

Where I_i is the ith integral term in equation (25).

Now, let there be a small functional variation δw, $\delta\zeta$ in $w\,(z)$ and $\zeta\,(z)$ respectively and compatible with the boundary conditions (20). Let δn be the corresponding change in n. We shall assume that δw, $\delta\zeta$ and δn are small enough to neglect their squares, products and higher powers. If δI_i, denote the corresponding variation in I_i then equation (26), give

$$-\,\delta n\,[2nI_1 + 2nI_3] = n^2 \delta I_1 + \frac{\mu k_x^2}{4\pi}\, \delta I_2 + n^2 \delta I_3 + \frac{\mu k_x^2}{4\pi}\, \delta I_4 - gk^2 \delta I_5 \quad (27)$$

where $$\delta I_1 = 2 \int_0^d \rho_0 \left[Dw D\delta w + k^2 w \delta w \right] = -\,2 \int_0^d \left[D\,(\rho_0 Dw) - k^2 \rho_0 w \right] \delta w$$

$$\delta I_2 = -\,2 \int_0^d \left[DH_0^2\, Dw + H_0^2\, (D^2 - k^2)\, w \right] \delta w$$

$$\delta I_3 = 2 \int_0^d \rho_0\, \zeta \delta\zeta \quad ; \quad \delta I_4 = 2 \int_0^d H_0^2\, \zeta \delta \zeta$$

$$\delta I_5 = 2 \int_0^d D\rho_0 w\, \delta w$$

Substituting δI_i in (27) and utilising (18) and (19), we get

$$-\delta n\left[nI_1+nI_3\right]=\int_0^d 2\Omega n D(\rho_\circ\zeta)\,\delta w+\int_0^d 2\Omega n\rho_\circ\,Dw\,\delta\zeta \quad .. \quad (28)$$

From (19) it can easily be deduced that

$$D(\delta w)=\frac{\delta\zeta\left[\rho_\circ n^2+\frac{\mu k^2_x}{4\pi}H_\circ^{\,2}\right]+\delta n\left[2n\rho_\circ\zeta-2\Omega\rho_\circ\,Dw\right]}{2\Omega\,\rho_\circ n} \quad .. \quad (29)$$

Integrating by parts the 1st term on the right hand side of (28) and utilising (29), we obtain

$$-\delta n\left[nI_1+nI_3+\int_0^d\zeta(2\Omega\rho_\circ Dw-2n\rho_\circ\zeta)\right]$$

$$=\int_0^d\delta\zeta\left[2\Omega n\rho_\circ Dw-n^2\rho_\circ\zeta-\frac{\mu k^2_x}{4\pi}H_\circ^{\,2}\zeta\right] \quad .. \quad .. \quad (30)$$

The bracket under the integral sign on the right hand side of (30) is zero because of (19) and since

$$\left[nI_1+nI_3+\int_0^d\zeta(2\Omega\rho_\circ Dw-2n\rho_\circ\zeta)\right]\neq 0 \quad \text{in general}$$

$$\delta n\equiv 0 \quad .. \quad .. \quad .. \quad . \quad .. \quad (31)$$

Hence we have a variational principle.

Rotation about x-axis.

In this case the linearised form of all the equations remain same except that of (8) which gives

$$\left.\begin{aligned}
\rho_\circ nu &= -ik_x\,\delta\rho+\frac{\mu}{4\pi}h_z\,DH_\circ \quad .. \quad .. \quad (i)\\
\rho_\circ nv-2\Omega\rho_\circ w &= -ik_y\,\delta\rho+\frac{\mu H_\circ}{4\pi}\left[ik_x h_y-ik_y h_x\right] \quad (ii)\\
\rho_\circ nw+2\Omega\rho_\circ v &= -D\delta\rho-g\delta\rho+\frac{\mu}{4\pi}\left[H_\circ(ik_x h_z-Dh_x)-h_z\,DH_\circ\right] \quad (iii)
\end{aligned}\right\}(32)$$

where $\nu_k = (1,0,0)$ 14

Following the procedure adopted in the last case, we can derive

$$\left(\rho_\circ n k^2 - \frac{gk^2}{n} D\rho_\circ \right) w - nD\,(\rho_\circ Dw) - 2\Omega i k_y\ wD\rho_\circ$$

$$- 2\Omega i k_x\ \rho_\circ \zeta = \frac{\mu k^2_x}{4\pi n}\left[DH_\circ^2 Dw + H_\circ^2 (D^2 - k^2)\, w \right] \quad (33)$$

and

$$\rho_\circ n\zeta + \frac{\mu k^2_x}{4\pi} H_\circ^2 \zeta - 2\Omega \rho_\circ\, i k_x\, w = 0 \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

From equations (33) and (34) with the help of (20) it can easily be shown that a variational principle can only exist if

$$k_x = 0 \quad (35)$$

i.e. if the disturbance is periodic in the y-direction only. This removes the effect of magnetic field on the resulting motion which is in the horizontal rolls with their axis along the x-axis. Thus within the scope of hydromagnetics, which is mainly concerned with the effect of magnetic field on fluid motions, this problem can not be based on a variational method.

PRINCIPLE OF EXCHANGE OF STABILITIES

If we take n_i and w_i to be complex and n_j and w_j to be their complex conjugates respectively, we can obtain from (24)

$$Im\,(n) \left[I_1 - I_3 + \frac{1}{|n|^2}\left(\frac{\mu k_x^2}{4\pi} I_4 + gk^2 I_5 - \frac{\mu k_x^2}{4\pi} I_2 \right) \right] = 0 \ (i)$$

and

$$Re\,(n) \left[I_1 + I_3 + \frac{1}{|n|^2}\left(\frac{\mu k_x^2}{4\pi} I_2 + \frac{\mu k_x^2}{4\pi} I_4 - gk^2 I_5 \right) \right] = 0 \ (ii) \quad (36)$$

An examination of equations (36) will show that either n is real or purely an imaginary quantity. In the latter case the characteristic value of n^2 will always be negative which means stability. This impiles that even if oscillatory modes exist, they are stable. In other words over-stability cannot arise.

This in fact is the case which arises in the two special problems considered in the next section.

Continuously stratified fluid of finite depth

We shall now consider the stability of a continuously stratified layer of fluid confined between two free horizontal surfaces $z=0$ and $z=d$ and rotating about z-axis. The permanent magnetic field is taken to be in the x-direction and stratified in the z-direction. Following Rayleigh we shall assume the density stratification law to be

$$\rho_\circ = \rho_1\, e^{\beta z} \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where ρ_1 and β are consistants. Again, following Talwar we shall assume that the local hydromagnetic velocity is constant. Thus

$$H_\circ^2/\rho_\circ = \text{const} = H_1^2/\rho_1$$

$$\text{or} \quad H_\circ^2 = H_1^2 \, e^{\beta z} \qquad (38)$$

where H_1^2 is constant.

The trial function satisfying the boundary conditions (20-*i*, *iii*) is assumed to be

$$w = A \sin s\pi z/d = A \sin pz \quad , \quad \text{where} \quad p = s\pi/d$$

From (19)

$$\zeta = \frac{2\Omega Apn}{n^2 + b_1^2 k_x^2} \cos pz = B \cos pz$$

where $B = 2\Omega Apn \,/\, (n^2 + b_1^2 \, k^2_x\,)$; $\quad b_1^2 = \mu H_1^2/4\pi\rho_1$

substituting from (39) and (40) in (25), utilising (37) and (38) and performing the integration, we have after some simple manipulation

$$n^4\,(1+\lambda) + n^2\,[2b_1^2 k^2_x\,(1+\lambda) - g\beta\lambda + 4\Omega^2 p^2] + b_1^4 k^4_x\,(1+\lambda) - g\beta\lambda b^2_1 k_x^2 = 0 \quad (41)$$

where $\lambda = 2k^2 \,/\, (2p^2 + \beta^2)$

Equation (41) is a quadratic in n^2 and may be written as

$$A_1 n^4 + B_1 n^2 + C_1 = 0 \qquad (42)$$

or

$$n^2 = \frac{-B_1 \pm \sqrt{B_1^2 - 4A_1 C_1}}{2A_1} \qquad (43)$$

It can be shown that the roots of (42) are real which indicates that over-stability is absent. While discussing the stability of the system we shall be interested in the unstable density stratification *viz.* $\beta > 0$. From (43) a necessary condition for instability is

$$C_1 < 0$$

$$\text{or} \quad b_1^2 < 2g\beta k^2 \,/\, k_x^2\,[\,2\,(p^2 + k^2) + \beta^2\,]$$

Thus a sufficient condition for stability is

$$b_1^2 = \frac{\mu H_1^2}{4\pi\rho_1} > \frac{2g\beta k^2}{k^2_x\,[\,2\,(p^2 + k^2) + \beta^2} \qquad (44)$$

It may be pointed out that in the absence of magnetic field this condition is never satisfied. Thus a magnetic field has a stabilizing influence on the fluid motion. From (44) it can be seen that the condition for stability is independent of rotation.

Equilibrium of two superposed fluids of great depths.

In this case the density configuration is

$$\rho_o(z) = \begin{cases} \rho_1 & 0 > z > -\infty \\ \rho_2 & 0 < z < \infty \end{cases} \quad \ldots \quad (45)$$

where ρ_1 and ρ_2 are constants. We shall assume that the magnetic field is uniform but is still in the horizontal direction. Equations (18) and (19) then combine to give

$$(n^2 + b^2k_x^2)^2 (D^2 - k^2) w + 4\Omega^2 n^2 D^2 w = 0 \quad \ldots \quad (46)$$

the boundary conditions are

$$\left.\begin{aligned} w_1(-\infty) &= w_2(+\infty) = 0 \\ \zeta_1(-\infty) &= \zeta_2(+\infty) = 0 \\ w_1(0) &= w_2(0) \\ [h_z(0)]_1 &= [h_z(0)]_2 \end{aligned}\right] \quad \ldots \quad (47)$$

and δp is continuous at the physical interface.

The suffixes 1 and 2 refer to the lower and upper fluids respectively. The solution of (46) satisfying the boundary conditions (47) is

$$\left.\begin{aligned} w_1 &= Ae^{\alpha z}; \quad w_2 = Ae^{-\alpha z} \\ \text{and } \zeta_1 &= C_1 e^{\alpha z}; \quad \zeta_2 = C_2 e^{-\alpha z} \end{aligned}\right] \quad \ldots \quad (48)$$

$$\text{where } \alpha = k\left[1 + \frac{4\,\Omega^2 n^2}{(n^2 + b^2 k_x^2)^2}\right]^{-1/2}$$

$$C_1 = \frac{2\,\Omega\, A\, a\, n}{n^2 + b_1^2 k_x^2}\;; \quad C_2 = \frac{-2\,\Omega\, A\, a\, n}{n^2 + b_2^2 k_n^2}$$

$$b_1^2 = \frac{\mu H_o^2}{4\pi\rho_1}\;; \quad b_2^2 = \frac{\mu H_o^2}{4\pi\rho_2}$$

Now ρ_o is constant everywhere except in the small region $\epsilon > z > -\epsilon$ where the transition from ρ_1 to ρ_2 takes place. Also we sub-divide the range of integration into 3 parts (following Hide[4]) *viz* $\infty > z > \epsilon$; $\epsilon > z > -\epsilon$ and $-\epsilon > z > -\infty$ and ultimately proceed to the limit $\epsilon = 0$.

Substitute for $w(z)$ and $\zeta(z)$ in (26), we have after performing the integration

$$I_1 = \int_{-\epsilon}^{\epsilon} \rho_o \left[(D\,w)^2 + k^2 w^2 \right] + A^2 (k^2 + d^2) \left[\rho_2 \int_{\epsilon}^{\infty} e^{-2\alpha z}\, dz + \rho_1 \int_{-\infty}^{-\epsilon} e^{2\alpha z}\, dz \right].$$

The contribution of the 1st vanishes in the limit since $\rho_o(z)$ is finite and w continuous. Thus

$$I_1 = \frac{A^2 (k^2 + a^2)}{2a} (\rho_1 + \rho_2) \quad ; \quad I_2 = H^2{}_o\, A^2 (k^2 + a^2)/a$$

$$I_3 = \frac{C_2{}^2 \rho_2 + C_1{}^2 \rho_1}{2a} \quad ; \quad I_4 = H^2{}_o\, (C_1{}^2 + C_2{}^2)/\, 2a$$

$$I_5 = \int_{\rho_1}^{\rho_2} w^2\, d\rho_o = A^2(\rho_2 - \rho_1)$$

where A^2 is the mean value of w^2 at $z = 0$. Substituting in (26), we get

$$[N_1 \rho_1 + N_2 \rho_2 - \lambda (\rho_2 - \rho_1)]\; N_1{}^2 N_2{}^2 + \chi^2 [N_1{}^3 \rho_1 + N_2{}^3 \rho_2] = 0 \tag{49}$$

where $N_1 = (n^2 + b_1{}^2 k^2{}_x)$; $N_2 = (n^2 + b_2{}^2 k^2{}_x)$

$$\chi^2 = \frac{4\, \Omega^2\, a}{k^2 + a^2} \quad ; \quad \lambda = 2\, g\, a\, k^2/(k^2 + a^2)$$

From (49)

$$\rho_1 N_1 \left(\frac{N_2{}^2 + \chi^2}{N_2{}^2}\right) + \rho_2 N_2 \left(\frac{N_1{}^2 + \chi^2}{N_1{}^2}\right) = \lambda\, (\rho_2 - \rho_1) \quad \cdot\cdot \tag{50}$$

Let $L = (N_2{}^2 + \chi^2) / N_2{}^2$; $M = (N_1{}^2 + \chi^2) / N_1{}^2$

clearly L and M are both positive and greater than unity.

Thus (50) gives

$$n^2 [L \rho_1 + M \rho_2] = \lambda (\rho_2 - \rho_1) - \frac{\mu H_o{}^2}{4\pi} k^2{}_x (L + M)$$

(*i*) For $\rho_1 < \rho_2$, n^2 is negative which corresponds to stability against all disturbances.

(*ii*) For $\rho_2 > \rho_1$ (unstable case). A necessary condition for instability is

$$\frac{\mu H_o{}^2}{4\pi}\; k^2{}_x < \frac{\lambda (\rho_2 - \rho_1)}{(L + M)}$$

Let k_m be the value of k_x which makes (52) an equality, there shall be instability only

when

$$k_x < k_m$$

where

$$k^2{}_m = \lambda (\rho_2 - \rho_1) \,/\, \frac{\mu H_o{}^2}{4\pi} (L + M)$$

In the absence of rotation, $L = M = 1$, $a = k$

(53) reduces to Talwar's[2] result. When $\Omega \neq 0$, $(L + M) > 2$ which shows that in the presence of rotation the instability when $\rho_2 > \rho_1$ will set in at a lower wave number than in the absence of rotation.

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