A DOUBLE-ENDED QUEUING SYSTEM

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ABSTRACT

The double-ended queuing problem first discussed by Kendall[1] with constant mean rates of customers and taxis arriving in a Poisson stream is considered in this paper under the assumption, that the inter-arrival time of taxis is distributed according to general distribution and only finite capacity for taxis is available.

Introduction

Kendall [1] discussed the double-ended queuing problem in which customers and taxis arrive at a taxi stand in a Poisson streams with constant mean rates. It has been shown that the difference of queue length has an exponential distribution with mean $(\lambda - \mu)$. Recently Dobbie[2] studied this problem with the parameters λ and μ as functions of time and obtained the probabilities that the difference of the queue lengths of taxis and customers attain a particular value at time t in terms of modified Bessel's function of the first kind. However in practice the taxi stand has got a finite capacity so that no more than N taxis can wait at a time. Therefore, a more realistic assumption seems to be that a taxi on finding N taxis already occupying the stand will have to go to other stand. Hence in this paper this double-ended queuing system has been examined under the condition that a taxi stand has a finite capacity for N taxis and inter-arrival distribution of taxis is general. This generalisation has been made to cover more general process of this type and also to examine the stochastic behaviour of this non-Markovian process.

Statement of the Problem

Customers arrive at a taxi stand in Poisson stream with mean rate λ and queue up if taxis are not available. Taxis arrive at this stand, the inter-arrival time distribution being general with probability density $S(x)$ and pick up one customer. It may be remarked that even if the customers arrive in group, as they do, the whole group may be taken as one unit, of course, assuming that a group does not consist of more persons than a taxi can admit. If a taxi arrives to find N taxis already waiting, it is forced to go away. In order to restore the Markovian character of the process we introduce a supplementary variable x, denoting the time since the last taxi arrived and define the following probability.

$p_n(x, t)\,dx$ denoting the probability that n units are waiting at time t and the last taxi arrived at a time between x and $x + dx$ earlier. The discrete variable n

(*i*) If positive, denotes the queue of taxis.

(*ii*) If zero, neither a taxi nor a customer is waiting.

(*iii*) If negative, customers are waiting.

Formulation of the Equation

Following Keilson and Kooharian[3] we relate the probability at time $t + \Delta$, to those at time t and obtain

$$p_n(x + \Delta, t + \Delta) = p_n(x,t)\left\{1 - (\lambda + \eta(x))\,\Delta\right\} + \lambda\Delta\, p_{n+1}(x,t) \qquad (-\infty \leqslant n < N) \qquad (1)$$

$$p_N(x + \Delta, t + \Delta) = p\,(x,t)\left\{1 - (\lambda + \eta(x))\,\Delta\right\} \quad .. \qquad (2)$$

where $\eta(x)$ is the first order probability that a taxi arrive in time x and $x + \Delta$ conditioned that it has not arrived upto x and is related to the probability density $S(x)$ by mean of relation

$$S(x) = \eta(x)\,\exp\left\{-\int_0^\infty \eta(x)\,dx\right\} \quad .. \quad .. \qquad (3)$$

as $\Delta \to 0$, equation (1) and (2) become

$$\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} + (\lambda + \eta(x))\,p_n(x,t) = \lambda\, p_{n+1}(x,t) \qquad (4)$$

$$\frac{\partial p_N(x,t)}{\partial t} + \frac{p_N(x,t)}{\partial x} + (\lambda + \eta(x))\,p_N(x,t) = 0 \quad .. \qquad (5)$$

These equations are to be solved subject to the boundary conditions.

$$p_n(o,t) = \int\int_0^\infty p_{n-1}(x,t)\,\eta(x)\,dx \quad .. \quad .. \qquad (6)$$

and

$$p_N(o,t) = \int_0^\infty \left[p_{N-1}(x,t) + p_N(x,t) \right] \eta(x)\,dx \qquad (7)$$

Solution of the Problem

Let us define the following generating function

$$f(x, a, t) = \sum_{n=-\infty}^{N-1} a^n\, p_n(x,t) \quad .. \quad .. \qquad (8)$$

Multiplying equation (4) by a^n and summing over n from $-\infty$ to N-1, we have

$$\frac{\partial f(x, a, t)}{\partial t} + \frac{\partial f(x, a, t)}{\partial x} + \left(\lambda + \eta(x) - \frac{\lambda}{a}\right) f(x, a, t) = \lambda\, a^{N-1}\, p_N(x,t) \quad .. \qquad (9)$$

and the boundary conditions become

$$f(o, a, t) = a \int_{o}^{\infty} f(x, a, t)\, \eta(x)\, dx - a^N \int_{o}^{\infty} p_{N-1}(x, t)\, \eta(x)\, dx \quad (10)$$

Let us suppose that the arrival of a taxi which makes the number of units in the system equal to K, where K<N, then

$$f(x, a, o) = a^K \delta(x) \quad \ldots \quad (11)$$

Where $\delta(x)$ is the Dirac delta function

Introducing the Laplace transform

$$\bar{f}(x, a, s) = \int_{o}^{\infty} e^{-st} f(x, a, t)\, dt$$

Similarly $\bar{p}_N(x, s)$ represents the Laplace transform of $p_N(x, t)$ and assuming the initial condition above, therefore the equation (9) becomes

$$\frac{\partial \bar{f}}{\partial x} + \left\{\lambda + s + \eta(x) - \frac{\lambda}{a}\right\} \bar{f} = \lambda a^{N-1} \bar{p}_N(x,s) + a^K \quad (12)$$

$$\frac{\partial \bar{p}_N(x,s)}{\partial x} + \left\{ \lambda + s + \eta(x) \right\} \bar{p}_N(x,s) = 0 \quad \ldots \quad (13)$$

$$\bar{f}(o, a, s) = a \int_{o}^{\infty} \bar{f}(x, a, s)\, \eta(x)\, dx - a^N \int_{o}^{\infty} \bar{p}_N(x,s)\, \eta(x)\, dx \quad (14)$$

and

$$\bar{p}_N(o,s) = \int_{o}^{\infty} \bar{p}_{N-1}(x,s)\, \eta(x)\, dx + \int_{o}^{\infty} \bar{p}_N(x,s)\, \eta(x)\, dx \quad (15)$$

Solution of (13) is given by

$$\bar{p}_N(x,s) = \bar{p}_N(o,s)\, e^{-(\lambda + s)x}\, e^{-\int_{o}^{x} \eta(x) dx} \quad \ldots \quad (16)$$

So that equation (11) becomes

$$\frac{\partial \bar{f}}{\partial x} + \left\{\lambda + s + \eta(x) - \frac{\lambda}{a}\right\} \bar{f} = \lambda a^{N-1}\, \bar{p}_N(o,s)\, e^{-(\lambda + s)x}\, e^{-\int_{o}^{x} \eta(x)\, dx} + a^K \quad \ldots \quad (17)$$

and the solution of (17) is

$$\bar{f}(x, a, s) = \left[a^K + \bar{f}(o, a, s) + a^N\, \bar{p}_N(o, s)\left(1 - e^{-\frac{\lambda}{a}x}\right)\right] \times e^{-\left(\lambda + s - \frac{\lambda}{a}\right)x}\, e^{-\int_{o}^{x} \eta(x) dx} \quad \ldots \quad (18)$$

Substituting this value of $\bar{f}(x, \alpha, s)$ in the boundary condition (14) and using (15), we get

$$\bar{f}(o, \alpha, s) = \frac{{}^{K+1}\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right) - \alpha^N \bar{p}_N(o, s)\left\{1 + \alpha\bar{S}(\lambda + s) - \bar{S}(\lambda + s) - \alpha\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right)\right\}}{1 - \alpha\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right)} \quad \text{(19)}$$

To obtain the value of $\bar{p}_N$ (o, s) we adopt the usual argument, but in this case the function $\bar{f}(o, \alpha, s) = \sum_{n=-\infty}^{N-1} \alpha^n \bar{p}_n(o, s)$ should be regular outside the unit circle $|\alpha| = 1$. It can be seen by putting $\alpha = \frac{1}{\beta}$ that the expression in the denominator of (19) has a unique zero lying outside the circle. Let it be α_o, then since the numerator must vanish on this value, we have

$$\bar{p}_N(o, s) = \frac{\alpha_o^{K-N}}{(\alpha_o - 1)\bar{S}(\lambda + s)} \quad \text{(20)}$$

Substituting this value in (18), we get

$$\bar{f}(o, \alpha, s) = \frac{\alpha^K\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right) + \frac{\alpha^N \alpha_o^{K-N}}{(\alpha_o - 1)\bar{S}(\lambda + s)}\left\{\alpha\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right) + \bar{S}(\lambda + s) - 1 - \alpha\bar{S}(\lambda + s)\right\}}{1 - \alpha\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right)} \quad \text{(21)}$$

and

$$f(x, \alpha, s) = \left[\alpha^K + \frac{\alpha^{K+1}\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right) + \frac{\alpha^N \alpha_o^{K-N}}{(\alpha_o - 1)\bar{S}(\lambda + s)}\left\{\alpha\bar{S}\left(\lambda + s - \frac{\lambda}{\alpha}\right) + \bar{S}(\lambda + s) - 1 - \alpha\bar{S}(\lambda + s)\right\}}{1 - \alpha S\left(\lambda + s - \frac{\lambda}{\alpha}\right)}\right.$$

$$\left. + \frac{\alpha^N \alpha_o^{K-N}}{(\alpha_o - 1)\bar{S}(\lambda + s)}\left(1 - e^{-\frac{\lambda}{\alpha}x}\right)\right] e^{-\left(\lambda + s - \frac{\lambda}{\alpha}\right)x} e^{-\int_0^x \eta(x)dx} \quad \text{(22)}$$

Hence the Laplace transform $\bar{\pi}(a, s)$ of probability generating function $\pi(a, t)$, is given by

$$\bar{\pi}(a, s) = \int_0^\infty \bar{f}(x, a, s)dx + a^N \int_0^\infty \bar{p}_N(x, s)dx$$

$$= \frac{a^K}{\lambda + s - \frac{\lambda}{a}} \quad \frac{1 - \bar{S}\left(\lambda + s - \frac{\lambda}{a}\right)}{1 - a\bar{S}\left(\lambda + s - \frac{\lambda}{a}\right)}$$

$$+ \frac{(1 - a)a^N a_0^{K-N}}{(a_0 - 1)} \quad \frac{1 - \bar{S}\left(\lambda + s - \frac{\lambda}{a}\right)}{\lambda + s - \frac{\lambda}{a}} \quad \ldots \quad (23)$$

For exponential case it comes.

$$\bar{\pi}(a, s) = \frac{a^K}{\mu - \mu a + s + \lambda - \frac{\lambda}{a}}$$

$$+ \frac{(1 - a)a^N a_0^{K-N}}{(a_0 - 1)} \quad \frac{1}{\mu + s + \lambda - \frac{\lambda}{a}}$$

$$= \frac{a^K}{\mu - \mu a + s + \lambda - \frac{\lambda}{a}} + \frac{(1 - a)\, a^N}{\mu - \mu a + s + \lambda - \frac{\lambda}{a}} \sum_{m=0}^{\infty} \beta_0^{N+1-K+m} \quad (24)$$

where

$$\beta_0 = \frac{(\mu + \lambda + s) + \sqrt{(\mu + \lambda + s)^2 - 4\lambda\mu}}{2\mu} \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Inverting (24), we have the probability generating function

$$\pi(a, t) = a^K e^{\left[\mu(1 - a) + \lambda\left(1 - \frac{1}{a}\right)\right]t}$$

$$+ (1 - a)a^N e^{-\left(\mu + \lambda - \frac{\lambda}{a}\right)t} \int_0^t e^{\frac{\lambda}{a}\tau} \sum_{m=0}^{\infty} \left(\sqrt{\frac{\lambda}{\mu}}\right)^{N+1-K+m}$$

$$\times I_{N+1-K+m}(2\sqrt{\lambda\mu}\,\tau)d\tau \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

where $I_n(x)$ is the modified Bessel function of the first kind.

Conclusions

The time dependent probability generating function for a double-ended queuing system under limited waiting space has been obtained. This study is useful in designing the waiting space for vehicles used by public for transportation.

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References

1. Kendall, D. G., *J. Roy, Stat. Soc.*, B **13**, 151, 1951.
2. Dobbie, J. M., *Opns. Res.*, **9**, 755, 1961.
3. Keilson, J., and Kooharian, A., *Ann. Math. Stat.*, **31**, 104, 1960.